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EFFICIENT INTERTEMPORAL ALLOCATION, CONSUMPTION-VALUE MAXIMIZATION AND CAPITAL-VALUE TRANSVERSALITY: A UNIFIED VIEW*

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I. INTRODUCTION AND SUMMARY

In recent years, based on Malinvaud's seminal work [15,16], there have been a number of important contributions [1,2,3,6,12,14,17, 18,19,20] towards resolving the following fundamental question: What observable properties characterize[1] some (all) competitive price system(s) associated with an efficient growth path? The natural focus in efforts to answer this question has been on the dynamical behavior of capital-value -- "natural" in the sense that Malinvaud's theorem establishes that, under very general conditions, efficient growth is tantamount to the existence of an associated competitive price system with the special property that, in every period, capital-value is minimized among all potential growth paths yielding the same stream of consumption goods thereafter. In other words, dealing with the question eventually reduces to looking for a more concrete version of Malinvaud's capital-value minimization condition.

Roughly speaking, reported results have come from two distinct approaches: First, from searching for particular productivity and substitution properties of technology which necessarily require asymptotically insignificant capital-value, or capital-value transversality (a label suggested by the common reference to the "transversality condition," especially in optimal growth theory) [1,12,14,18,19], and second, from searching for weaker regularities in the asymptotic behavior of capital-value which necessarily emerge given particular productivity and substitution properties of technology [1,2,3,6,20]. The first approach is undoubtedly more appealing intuitively, since -- at least when there are no primary factors -- capital-value transversality

is essentially equivalent to consumption-value maximization, i.e., the property that among all feasible growth paths, and at its associated consumption goods prices, an efficient growth path maximizes the total value of the whole stream of consumption goods. Unfortunately, this approach encounters great difficulty from the mere presence of primary factors[2] (and the second approach from the mere presence of polyhedral-like technology). Nonetheless, it is basically the approach we adopt here. Thus, one of our purposes in this paper is to present general conditions on static productivity characteristics and substitution possibilities which verify the property of consumption-value maximization, and incidentally, capital-value transversality as well (in Section V and Appendices A and B).

A closely related result appears in recent contributions concerning the problem of how to characterize optimal growth paths [23,24,25,26,32]: Suppose that we are given some nonnegative, nontrivial values for the whole stream of consumption goods.[3] If, at these given consumption goods values, a particular feasible growth path yields maximum total value among all feasible growth paths, i.e., is an optimal growth path, then it has an associated competitive price system with the special properties that (i) consumption goods prices are the same as the given values, and (ii) capital-value transversality obtains. In short, the proposition is simply that consumption-value maximization (at given consumption goods values) implies capital-value transversality (at associated capital goods prices). Another of our purposes in this paper is to present a simpler proof for this proposition (Section VI).

Perhaps our principal purpose, however, is to display the common structure underlying both consumption-value maximization in efficient growth theory and capital-value transversality in optimal growth theory. The mathematical tool we use to establish this connection is the basic support theorem implicitly developed in the Majumdar-Mitra-McFadden proof of consumption-value maximization for the closed, multi-sector model of growth [14] (see also its precursors [13,17]), and this unifying result is presented first (in Section II). (The reader who is primarily interested in our economic applications can skim or skip this section, which is unavoidably technical though quite straightforward.) Then, after describing our general growth model (in Section III), we turn to characterizing efficient and optimal growth paths (in Sections IV-VI), and to elaborating the complications arising from the presence of primary factors(in Sections VII-VIII). In the course of accomplishing our avowed purposes, we also develop two subsidiary results of interest in their own right: First, we present an alternative argument leading to Malinvaud's theorem, one which has the additional virtue of introducing a broader sense in which efficient growth paths yield value maximization, and hence also capi minimization and consumption-value maximization (in Section IV). (See too the various arguments in [12,22,30].) Second, we present a general theorem establishing the existence of special competitive price systems associated with feasible growth paths which are optimal in various weaker senses than that of simple consumption-value maximization (at given consumption goods values; in Section VIII). (See too the related theorems in [9,27,28,29].)

II. BASIC SUPPORT THEOREM

We use $x = (x_0, x_1, \ldots)$ and $\pi = (\pi_0, \pi_1, \ldots)$ to denote sequences (later on x will be interpreted as quantities, π as prices), ℓ_1 to denote the space of all summable sequences with sum norm $||x||_1 = \sum_{t=0}^{\infty} |x_t|$, and ℓ_∞ to denote the space of all bounded sequences with sup norm $||x||_\infty = \sup_t |x_t|$. Also, $\pi \cdot x$ means $\sum_{t=0}^{\infty} \pi_t x_t$, $x' \geqq x''$ means $x'_t \geqq x''_t$ for every t, $x' \geq x''$ means $x'_t \geqq x''_t$ for every t and $x'_t > x''_t$ for some t, and $x' > x''$ means $x'_t > x''_t$ for every t.

Support Theorem (Majumdar-Mitra-McFadden): Suppose $X \subset \ell_1$ is closed, convex, freely disposable (i.e., if $x \in X$, $x' \in \ell_1$ and $x' \leqq x$, then $x' \in X$) and uniformly "productive" (i.e., there exists $\delta > 0$ such that for every $t \geqq 0$ there exists $x^t \in X$ such that $x^t_s \geqq \delta$ for $s = t$ and $x^t_s \geqq 0$ for $s \neq t$). If x^* is a boundary point of X (i.e., $x^* \in X$ but $x^* \notin$ interior X), then there exists $\pi \in \ell_\infty$ such that $\pi \geq 0$ and

(1) $$\pi \cdot x^* \geqq \pi \cdot x \text{ for every } x \in X.$$

Proof: Our argument is virtually identical to the original argument in [14], but since that argument is imbedded in a fairly long, model-oriented proof, we detail it here for convenience' sake. We will use two basic results from analysis:

1. (Separation of convex sets by a continuous linear functional.) If C^1 and C^2 are nonempty, convex subsets of a topological (e.g., normed) linear space S, interior $C^1 \neq \phi$ and interior $C^1 \cap C^2 = \phi$, then there exists a nontrivial, continuous linear functional f on S such that

$$f(c^2) \geqq f(c^1) \text{ for every } c^1 \in C^1,\ c^2 \in C^2.$$

2. (Representation of continuous linear functionals on ℓ_1.) If f is a continuous linear functional on ℓ_1, then there exists $\pi \in \ell_\infty$ such that

$$f(x) = \pi \cdot x \text{ for every } x \in \ell_1.$$

(See, for example, [10], especially Section 14 and exercises.)

Given these results, the idea of the proof is quite simple: Verify the hypotheses of the separation theorem when $S = \ell_1$, $C^1 = X$ and $C^2 = \{x^*\}$, and then apply the representation theorem.

Two of the hypotheses required for applying the separation theorem are easily verified (since X is assumed a nonempty, convex subset of ℓ_1, and x^* a boundary point of X). The remaining hypothesis -- that interior $X \neq \phi$ -- requires a little more work. What we need to show is that for some $x' \in X$ there exists $\varepsilon > 0$ such that if $||x - x'||_1 < \varepsilon$, then $x \in X$. In fact, we show that $x' = 0$ will do: By free disposal and uniform "productivity" $0 \in X$. Now pick $0 < \varepsilon \leqq \delta$ and suppose, without loss of generality, $x_0 \neq 0$ and $\|x - 0\|_1 = \|x\|_1 = \alpha < \varepsilon$. Then (again by free disposal and uniform "productivity"), if $x_t \geqq 0$ (<0),

then there exists $x^t \in X$ such that $x_s^t = \alpha$ $(-\alpha)$ for $s = t$ and $x_s^t = 0$ for $s \neq t$. Let $\lambda_s^t = |x_s| \,/\, \sum_{u=0}^{t} |x_u|$ for $0 \leqq s \leqq t$, so that $\lambda_s^t \geqq 0$ and $\sum_{s=0}^{t} \lambda_s^t = 1$, and let $z^t = \sum_{s=0}^{t} \lambda_s^t x^s$ for $t \geqq 0$, so that

$$z^t = (\alpha / \sum_{u=0}^{t} |x_u|)\ (x_0, x_1, \ldots, x_t, 0, \ldots).$$

Then (by convexity) $z^t \in X$ while (by closedness) if $\lim_{t\to\infty} z^t = x$, then $x \in X$. But

$$||z^t - x||_1 = \sum_{s=0}^{\infty} |z_s^t - x_s|$$

$$= \sum_{s=0}^{t} |(\alpha / \sum_{u=0}^{t} |x_u|) x_s - x_s| + \sum_{s=t+1}^{\infty} |x_s|$$

$$= (\alpha / \sum_{u=0}^{t} |x_u| - 1) \sum_{s=0}^{t} |x_s| + \sum_{s=t+1}^{\infty} |x_s|$$

$$= (\sum_{s=0}^{\infty} |x_s| - \sum_{s=0}^{t} |x_s|) + \sum_{s=t+1}^{\infty} |x_s|$$

$$= 2 \sum_{s=t+1}^{\infty} |x_s|,$$

that is, $\lim_{t\to\infty} z^t = x$.

Now applying the separation theorem, we know that there exists a nontrivial, continuous linear functional f on ℓ_1 such that $f(x^*) \geqq f(x)$ for every $x \in X$. But applying the representation theorem, we know that there exists $\pi \in \ell_\infty$ such that $f(x) = \pi \cdot x$ for every $x \in \ell_1$. Since f is nontrivial, $\pi \neq 0$, while since X is freely disposable, $\pi \geqq 0$.

Hence, $\pi \geq 0$, and the proof is complete. ∎

The key to the foregoing argument is the fact that the maintained assumptions on X are strong enough to guarantee that X has an interior point. Clearly weaker assumptions (for example, simply that X has an interior -- which obviously doesn't require that X be closed) will suffice for this purpose. However, there doesn't seem to us to be much gain in elaborating them, since in most economic applications (in particular, in those discussed below), weaker assumptions would be hard to verify in terms of the underlying economic model. Moreover, all but the assumption of uniform "productivity" are fairly conventional in most economic applications.

Finally, for several of our applications of the Support Theorem it is important to know that the assumption of uniform "productivity" can be replaced by the following pair of assumptions: Possibility of no production (i.e., $0 \in X$) and uniform productivity (i.e., there exist $\delta > 0$ and $t^{\delta} < \infty$ such that for every $t > t^{\delta}$ there exists x^t such that $x_s \geqq -\delta$ for $0 \leqq s \leqq t^{\delta}$, $x_s \geqq \delta$ for $t^{\delta} < s = t$ and $x_s \geqq 0$ for $t^{\delta} < s \neq t$). This follows from the easily verified fact that, if X satisfies these as well as the remaining maintained assumptions of the Support Theorem, then $X^{\delta} = X + (\overbrace{\delta,\delta,\ldots,\delta}^{t^{\delta}},0,\ldots)$ satisfies the full set of maintained assumptions of that theorem, including uniform "productivity."

III. GENERAL GROWTH MODEL

Feasible real allocations or growth paths in the economy are described by

$$(2)\quad \begin{cases} (c_t,z_t,k_t) \in T \text{ and } k_{t+1} = z_t \text{ for } t \geqq 0, \\ k_0 = \bar{k} > 0 \end{cases}$$

where $c = (c_1,c_2,\ldots,c_m)$ is an m-vector of consumption goods output, $z = (z_1,z_2,\ldots,z_n)$ is an n-vector of ("gross") capital goods, or better, investment goods output, $k = (k_1,k_2,\ldots,k_n)$ is an n-vector of capital stock inputs, $T = \{(c,z,k)\}$ is the static technology available for producing outputs from capital stock inputs,[4] t is an index representing discrete production periods (so that 0 is the initial period), and $\bar{k} = (\bar{k}_1,\bar{k}_2,\ldots,\bar{k}_n)$ is an n-vector of initial capital stocks. (The assumption that initial capital stocks are positive $\bar{k} > 0$ enters the analysis in a nontrivial way.)

Regarding the technology, we will always assume that

T is a nonnegative, closed set exhibiting

T1. Generalized Diminishing Returns: T is convex;

T2. Free Disposal: If $(c,z,k) \in T$, $0 \leqq (c',z') \leqq (c,z)$ and $k' \geqq k$, then $(c',z',k') \in T$;

and

T3. Constant Returns to Scale: T is a cone.[5]

The whole of our analysis depends critically on how productive inputs are (and also, to a lesser extent, on how substitutable outputs are). For the time being we will only postulate the following productivity characteristics:

T4. Necessity of Capital Stocks: If $(c,z,k) \in T$ and $k = 0$, then $(c,z) = 0$;

and

T5. Productivity of Capital Stocks: If $(c,z,k) \in T$ and $k' > k$, then there exists $(c',z') > (c,z)$ such that $(c',z',k') \in T$.

T4 is one representation of the idea of scarcity, T5 of the idea of (the productivity of) roundaboutness. Note, in particular, that T5 implies that positive outputs can be produced from positive capital stocks. Later on we will systematically enlarge on this fairly unrestrictive specification.

Corresponding to (2), the "feasible" value inputations or price systems in the economy are described by

$$(3) \quad \begin{cases} (p_t, q_t, r_t) \in M \text{ and } r_t = q_{t-1} \text{ for } t \geqq 0, \\ q_{-1} \geqq 0 \end{cases}$$

where $p = (p_1, p_2, \ldots, p_m)$ is an m-vector of consumption goods prices,

$q = (q_1, q_2, \ldots, q_n)$ is an n-vector of investment goods prices,

$r = (r_1, r_2, \ldots, r_n)$ is an n-vector of ("gross") capital stock rents, and

$M = \{(p,q,r) : (p,q) \geqq 0 \text{ and } p \cdot c + q \cdot z - r \cdot k \leqq 0 \text{ for every } (c,z,k) \in T\}$

is the nonnegative subset of the dual cone to T. (3) describes, among others, all the price systems which might be observed in a (perfect foresight) competitive equilibrium for the economy, and simply represents a convenient way of summarizing the possibilities for price systems (in exactly the same manner that (2) simply represents a convenient way of summarizing the possibilities for growth paths). We will refer to a particular feasible growth path (i.e., a particular solution to (2) denoted, say, by asterisks) as being competitive, or having an associated competitive price system, if there is some nontrivial "feasible" price system (i.e., some nontrivial solution to (3)) at which the given feasible growth path yields maximum profit of zero among all potential input-output combinations in each period,

$$(4) \quad p_t \cdot c + q_t \cdot z - q_{t-1} \cdot k \leqq p_t \cdot c_t^* + q_t \cdot z_t^* - q_{t-1} \cdot k_t^* = 0$$

for every $(c,z,k) \in T$ and $t \geqq 0$.

Of course, only some feasible growth paths are competitive, and only some competitive growth paths are efficient (or optimal), which is the underlying raison d'etre for this paper.

Before proceeding with the analysis, we emphasize two implications of (4) which we will use repeatedly: Consider a particular competitive growth path (denoted again by asterisks). Since if $k_0 = \bar{k}$ and $k_{t+1} = z_t$ for $t \geqq 0$, then

$$\sum_{t=0}^{\mathit{t}} p_t \cdot c_t = \sum_{t=0}^{\mathit{t}} (p_t \cdot c_t + q_t \cdot (z_t - k_{t+1}))$$

$$= \sum_{t=0}^{\mathit{t}} (p_t \cdot c_t + q_t \cdot z_t - q_{t-1} \cdot k_t) + q_{-1} \cdot \bar{k} - q_{\mathit{t}} \cdot k_{\mathit{t}+1} \text{ for } \mathit{t} \geqq 0,$$

it follows from (4) that

(5) $\sum_{t=0}^{\mathit{t}} p_t \cdot c_t \leqq \sum_{t=0}^{\mathit{t}} p_t \cdot c_t + q_{\mathit{t}} \cdot k_{\mathit{t}+1} \leqq q_{-1} \cdot \bar{k}$ for $\mathit{t} \geqq 0$ for every feasible growth path (2),

while

(6) $\sum_{t=0}^{\mathit{t}} p_t \cdot c_t^* \leqq \sum_{t=0}^{\mathit{t}} p_t \cdot c_t^* + q_{\mathit{t}} \cdot k_{\mathit{t}+1}^* = q_{-1} \cdot \bar{k}$ for $\mathit{t} \geqq 0$.

Among other things, (5) tells us immediately that consumption-value is uniformly bounded

$$\sum_{t=0}^{\infty} p_t \cdot c_t \leqq q_{-1} \cdot \bar{k} \text{ for every feasible growth path (2),}$$

so that (5) and (6) together tell us immediately that capital-value transversality implies consumption-value maximization

$$\lim_{t \to \infty} q_t \cdot k_{t+1}^* = 0, \quad \Rightarrow \infty > \sum_{t=0}^{\infty} p_t \cdot c_t^* \geqq \sum_{t=0}^{\infty} p_t \cdot c_t \text{ for every feasible growth path(2).}$$

(The converse to this latter proposition will be established in Section VI).

IV. CHARACTERIZATION OF EFFICIENCY

A particular feasible growth path (denoted once again by asterisks) is efficient if there is no other feasible growth path which dominates it in terms of the whole stream of consumption goods, i.e., if there is no other solution to (2) such that

(7) $(c_0, c_1, \ldots) \geq (c_0^*, c_1^*, \ldots)$.

This concept lies at the very core of (neoclassical) economics, since efficient growth paths constitute the subset of feasible growth paths which are indisputably "desirable" or "good."

In order to characterize such paths, we now interpret the quantity sequence x as

(8) $$x = (\overbrace{\bar{k} - k_0}^{(\bar{k} - k_0) - (\bar{k} - k_0^*)}, c_0 - c_0^*, z_0 - k_1, \ldots, c_t - c_t^*, \overbrace{z_t - k_{t+1}}^{(z_t - k_{t+1}) - (z_t^* - k_{t+1}^*)}, \ldots),$$

the price sequence π as

(9) $$\pi = (q_{-1}, p_0, q_0, \ldots, p_t, q_t, \ldots),$$

and the set X as

(10) $$X = \{x : x \in \ell_1, c_t \leqq c_t' \text{ and } (c_t', z_t, k_t) \in T \text{ for } t \geqq 0\}.$$

That is, x is taken to be the sequence of net final outputs (i.e., outputs today in excess of inputs tomorrow) over and above those generated by a particular efficient growth path, π the corresponding sequence of value imputations, and X the set of all summable x which could be produced given unlimited free disposal of consumption goods and unlimited costless availability of capital stocks. (It is basically the latter flexibility which finally gives economic content to application of the Support Theorem; a particular growth path used to generate a particular sequence of net final outputs need only be feasible in the special sense defined by (10), and not in the usual sense defined earlier by (2).)

Given these interpretations, it is relatively straightforward to verify that $X \subset \ell_1$ is closed,[7] convex and freely disposable by virtue of our maintained assumptions on T (exclusive of T5). Furthermore, the quantity sequence generated by the efficient growth path itself,

$$x^* = (\bar{k} - k_0^*, c_0^* - c_0^*, z_0^* - k_1^*, \ldots, c_t^* - c_t^*, z_t^* - k_{t+1}^*, \ldots) = 0,$$

must be a boundary point of X: Obviously, x^* is in X. Suppose it were an interior point as well. Then (appealing to the way points in X are defined) it would be possible to increase an arbitrarily chosen consumption good output (for example, $c_{10} - c_{10}^* > 0$), while at the same time maintaining every other consumption good output ($c_{i0} - c_{i0}^* = 0$ for $i \neq 1$ and $c_t - c_t^* = 0$ for $t \neq 0$) as well as feasibility ($\bar{k} - k_0 = 0, z_t - k_{t+1} = 0$ and $(c_t, z_t, k_t) \in T$ for $t \geqq 0$). But such a possibility contradicts the fact that we began by postulating an efficient growth path.

From these considerations (together with the final comment at the end of Section II), it follows that if we could only verify that X is uniformly productive, then we could simply apply the Support Theorem to derive a price characterization for all efficient growth paths. This is indeed the strategy we follow in the subsequent section (but for a simplification in the interpretation of the quantity sequence x). However, this line of attack requires more structure on T then we have thus far imposed (see for example T6 and T7 below). Here we proceed instead by noticing that, although X is not necessarily <u>uniformly</u> productive, by virtue of T5 it is certainly productive, in the following sense: There exists $(x_0', x_1', \ldots, x_{n-1}') < 0$ such that for $t \geqq n$ there exists $x^t \in X$ such that $x_s^t \geqq x_s'$ for $0 \leqq s < n$, $x_s^t > 0$ for $n \leqq s = t$ and $x_s^t \geqq 0$ for $n \leqq s \neq t$. This seemingly complicated statement is nothing more than a formalization of the fact that T5 implies that if there were larger initial capital stocks available, then it would be feasible to produce no smaller net final outputs in every subsequent period, and actually larger net final outputs in any given subsequent period. In particular, the quantity sequences x^t for $t \geqq n$ can be generated by a growth path which utilizes given additional initial capital stocks (so that $x_{j-1}' = \bar{k}_j - k_{j0} < 0$ for $1 \leqq j \leqq n$) to produce more investment goods output during periods 0 through s-1, or more capital stock inputs for periods 0 through s, and more of both consumption goods output and investment goods output during period s (so that $n + (n+m)s < t \leqq n + (n+m)(s+1)$).

The observation that X is productive immediately suggests the following trick to render it uniformly so: Choose new units for measuring net final outputs or quantities, say ν_t for $t \geqq 0$, in such a manner that for some $\delta > 0$ and $t^\delta = n-1$, $\nu_t x_t^{\prime} \geqq -\delta$ for $0 \leqq t \leqq t^\delta$ and $\nu_t x_t^t \geqq \delta$ for $t > t^\delta$.[8] Clearly, such a maneuver enables us to apply the conclusion of the Support Theorem (1), and thereby (referring to the interpretations (8)-(10)) to establish a fundamental proposition:[9]

Price Characterization of Efficient Growth Paths: If a feasible growth path is efficient, then there exist positive units for measuring outputs $(v_{-1}, u_0, v_0, \ldots, u_t, v_t, \ldots) > 0$ and nonnegative, nontrivial prices for evaluating outputs $(q_{-1}, p_0, q_0, \ldots, p_t, q_t, \ldots) \geq 0$ such that

$$(11) \qquad 0 \geqq q_{-1} \cdot (\bar{k} - k_0) + \sum_{t=0}^{\infty} (p_t \cdot (c_t - c_t^*) + q_t \cdot (z_t - k_{t+1}))$$

for every growth path $(c_t, z_t, k_t) \in T$ such that

$$\sum_{j=1}^{n} v_{j,-1} |\bar{k} - k_{j0}| + \sum_{t=0}^{\infty} \Big(\sum_{i=1}^{m} u_{it} |c_{it} - c_{it}^*| + \sum_{j=1}^{n} v_{jt} |z_{jt} - k_{jt+1}| \Big) < \infty.$$

The converse obtains provided that consumption goods prices are positive $p_t > 0$ for $t \geqq 0$.

In this form, our price characterization is neither very transparent nor very usable. Three immediate corollaries go part way toward remedying these defects:

Value Characterizations of Efficient Growth Paths: If a feasible growth path is efficient, then it has an associated competitive price system (4) at which

1. Restricted Value Maximization: Net-final-output-value is maximized,

$$(12) \qquad \infty > \sum_{t=0}^{\infty} p_t \cdot c_t^* - q_{-1} \cdot \bar{k} \geqq \sum_{t=0}^{\infty} p_t \cdot c_t - q_{-1} \cdot k;$$

2. Restricted Capital-Value Minimization: Capital-value is minimized,

$$(13) \qquad \infty > \sum_{t=0}^{\infty} p_t \cdot c_t \geqq (>) \sum_{t=0}^{\infty} p_t \cdot c_t^* \Rightarrow q_{-1} \cdot \bar{k} \leqq (\qquad \text{and}$$

3. Restricted Consumption-Value Maximization: Consumption-value is maximized,

$$(14) \qquad k \leqq \bar{k} \Rightarrow (q_{-1} \cdot k \leqq q_{-1} \cdot \bar{k} \Rightarrow) \ \infty > \sum_{t=0}^{\infty} p_t \cdot c_t^* \geqq \sum_{t=0}^{\infty} p_t \cdot c_t,$$

among all growth paths which (i) are feasible from nonnegative capital stocks, or satisfy (2) with $\bar{k} = k \geqq 0$ and (ii) yield sufficiently small deviations in the whole stream of consumption goods, or satisfy

$$\sum_{t=0}^{\infty} \sum_{i=1}^{m} u_{it} |c_{it} - c_{it}^*| < \infty$$

for some (fixed) positive units for measuring consumption goods $u_{it} > 0$ for $1 \leqq i \leqq m$, $t \geqq 0$. In each case 1-3, the converse obtains provided that consumption goods prices are positive $p_t > 0$ for $t \geqq 0$.

Proof: (4) follows from (11) with $(c_s, z_s, k_s) = (c, z, k) \in T$ for $s = t$ and $(c_s, z_s, k_s) = (c_s^*, z_s^*, k_s^*)$ for $s \neq t$. Then, in view of (5), (12)-(14) follow from (11) when comparison is further limited to growth paths which are feasible from nonnegative capital stocks $k \geqq 0$. ∎

"Restricted" here refers to the fact that, among growth paths which are feasible from nonnegative capital stocks, we have further confined comparisons to just those which yield sufficiently small deviations in the whole stream of consumption goods. Notice, on the other hand, that we could have expanded such restricted comparisons to the class of growth paths which are identical up to period t, $(c_s,z_s,k_s) = (c^*_s,z^*_s,k^*_s)$ for $0 \leqq s < t$, and then are feasible from nonnegative capital stocks in period t, $k_t = k \geqq 0$, and $(c_s,z_s,k_s) \in T$ and $k_{s+1} = z_s$ for $s \geqq t$, for arbitrary distinguished period $t \geqq 0$. (This expansion would permit a more general interpretation of capital goods prices, as detailed by one of us elsewhere [8]).

Regarding the various value characterizations themselves: Restricted value maximization is the closest analogue of the standard characterization of efficient allocation in a static (or equivalently, dynamic but finite) economy, a parallel also elaborated at length in [8]. Restricted capital-value minimization is essentially a stronger version of Malinvaud's theorem, since (in light of the comments above) our condition involves a broader class of comparisons than does his (though at the cost mentioned in footnote 7). Finally, restricted consumption-value maximization basically speaks for itself. One of our prime goals is to sharpen this particular implication of (11).

Specifically, one can ask whether there are circumstances under which consumption-value maximization obtains without any restriction except feasibility. Alternatively, one can ask whether there are circumstances under which some more qualified notion of

optimality obtains without any restriction except feasibility (as is the case, for example, in the one-good model analyzed by Cass-Yaari [5]). It is to the former question we now turn; we will also respond, somewhat indirectly, to the latter question later on (in Sections VII and VIII, and at the end of Appendix B).

Before addressing this question, however, we emphasize at this point that each of our subsequent arguments amounts to a variant of the argument presented in this section: Namely, in each specific case, we interpret x,π,X and x^* in such a way that direct application of the Support Theorem plus judicious choice of comparison sequences leads to the desired conclusion.

V. EFFICIENCY AND CONSUMPTION-VALUE MAXIMIZATION

The argument in the preceding section underlines the basic difficulty involved in using the Support Theorem as a tool for characterizing efficient growth paths: The requirements of summability (i.e., $X \subset \ell_1$) and interiority (i.e., interior $X \neq \phi$) generally reflect conflicting aspects of the productive capacities of the underlying technology. Thus, a particular growth model in which feasible growth paths necessarily yield a summable stream of consumption goods will typically be incapable of providing consumption goods at a uniform rate in any given future period, and vice-versa. This trade-off is most clearly illustrated by the fact that, for the general growth model we are considering, by choosing units for measuring consumption goods outputs which embody a sufficiently high rate of discount we can guarantee summable consumption possibilities, while by choosing units which embody a sufficiently low rate we can guarantee uniform consumption possibilities, but usually not both.

There is one class of growth models, however, in which these conflicting aspects just balance each other. This class consists of the growth models where technology exhibits the following, additional productivity characteristics:

T6. Impossibility of Storage with Sustenance: There exist $p > 0$ and $q \geqq 0$ such that

$$p \cdot c + q \cdot z - q \cdot k \leqq 0 \text{ for every } (c,z,k) \in T;$$

and

T7. Possibility of Storage and Regeneration: There exist $0 \leq \hat{k} \leq \bar{k}$ and $\hat{z} > 0$ such that both $(0,\hat{k},\hat{k}) \in T$ and $(0,\hat{z},\hat{k}) \in T$.

T7 is more or less self-explanatory. It simply states that the economy is capable of replicating some capital stocks from which it is also capable of producing positive investment goods output -- or together with T5, positive outputs of both consumption and investment goods) within at most two periods. T7 would be true, for instance, if the economy were capable of replicating positive capital stocks, i.e., if there exists $0 < \hat{\hat{k}} \leqq \bar{k}$ such that $(0,\hat{\hat{k}},\hat{\hat{k}}) \in T$ (taking $\hat{k} = \hat{z} = \hat{\hat{k}}$ in T7).

T6, however, requires a bit more elaboration. Its label was chosen because it implies (and is implied by) the productivity limitation that the (closure of the) set of timeless potential net outputs

(15) $N = \{(c,y) : y = z - k \text{ and } (c,z,k) \in T\}$

does not contain any (c',y') such that both $c' \geq 0$ and $y' \geqq 0$. In other words, roughly speaking, T6 amounts to the assumption that the economy is not capable of replicating some capital stocks while simultaneously producing some consumption goods output. In Appendix A we establish this equivalency, and present two alternative conditions on technology -- one involving substitution properties of outputs, the other productivity properties of exhaustible resource inputs -- which entail T6 (and which are more interpretable, if more restrictive).

In any case, for the class of growth models where the technology satisfies these two assumptions (as well as our other maintained assumptions) we have a slightly generalized version of what might (from our present perspective) be viewed as the central result in [14]:

Consumption-Value Maximization: Suppose T satisfies T6 and T7. If a feasible growth path is efficient, then it has an associated competitive price system (4) at which consumption-value is maximized,

(16) $$\infty > \sum_{t=0}^{\infty} p_t \cdot c_t^* \geqq \sum_{t=0}^{\infty} p_t \cdot c_t,$$

among all feasible growth paths (2). The converse obtains provided consumption goods prices are positive $p_t > 0$ for $t \geqq 0$.

Proof: We now interpret the quantity sequence x as simply net final outputs

(17) $\quad x = (\bar{k} - k_0, c_0, z_0 - k_1, \ldots, c_t, z_t - k_{t+1}, \ldots),$

but the price sequence π and set X as before in (9) and (10), respectively. The crucial steps in the proof are first (summability), establishing that by virtue of T6,

(18) $\quad \sum_{t=0}^{\infty} \sum_{i=1}^{m} c_{it} < \infty$ for every solution to (2),

so that the set X contains every sequence of net final outputs which is generated by a feasible growth path, and second (interiority), establishing that by virtue of T7, the set X is uniformly "productive." The remainder of the argument then simply consists in applying and interpreting the Support Theorem.

To see that (18) follows from T6, we use the stationary competitive price system (p,q) to evaluate the whole stream of consumption goods produced along an arbitrary feasible growth path:

$$\sum_{t=0}^{\mathit{t}} \sum_{i=1}^{m} (\min_i\{p_i\}) c_{it} \leqq \sum_{t=0}^{\mathit{t}} p \cdot c_t = \sum_{t=0}^{\mathit{t}} (p \cdot c_t + q \cdot (z_t - k_{t+1})) =$$

$$\sum_{t=0}^{\mathit{t}} (p \cdot c_t + q \cdot z_t - q \cdot k_t) + q \cdot \bar{k} - q \cdot k_{\mathit{t}+1} \leqq q \cdot \bar{k} \text{ for } \mathit{t} \geqq 0.$$

An immediate consequence of this value-bound is

$$\sum_{t=0}^{\mathit{t}} \sum_{i=1}^{m} c_{it} \leqq q \cdot \bar{k} \Big/ \min_i\{p_i\} < \infty \text{ for } \mathit{t} \geqq 0,$$

and hence (18).

To see that uniform "productivity" follows from T7 (given

our maintained assumptions about initial capital stocks and technology) note first that T5 implies that (i) there exists $(\bar{c}', \bar{z}') > 0$ such that $(\bar{c}', \bar{z}', \bar{k}) \in T$ and (ii) there exists $(\hat{c}', \hat{z}') > 0$ such that $(\hat{c}', \hat{z}', \hat{z}) \in T$. Hence, the following quantity sequences $x \in X$ are generated by the corresponding growth paths $(c_s, z_s, k_s) \in T$ for $s \geqq 0$:

$$x = (\bar{k}, 0, \ldots) \quad \text{when } (c_s, z_s, k_s) = 0 \quad \text{for } s \geqq 0$$

$$= (0, \bar{c}', \bar{z}', 0, \ldots) \quad = \begin{cases} (\bar{c}', \bar{z}', \bar{k}) & \text{for } s = 0 \\ 0 & \text{for } s \geqq 1 \end{cases}$$

$$= (k - \hat{k}, 0, 0, \hat{c}', \hat{z}', 0, \ldots) \quad = \begin{cases} (0, \hat{z}, \hat{k}) & \text{for } s = 0 \\ (\hat{c}', \hat{z}', \hat{z}) & \text{for } s = 1 \\ 0 & \text{for } s \geqq 2 \end{cases}$$

and

$$= (k - \hat{k}, 0, \ldots, 0, \hat{c}', \hat{z}', 0, \ldots) \quad = \begin{cases} (0, \hat{k}, \hat{k}) & \text{for } 0 \leqq s \leqq t - 2 \\ (0, \hat{z}, \hat{k}) & \text{for } s = t - 1 \\ (\hat{c}', \hat{z}', \hat{z}) & \text{for } s = t \\ 0 & \text{for } s \geqq t + 1 \end{cases}$$

for $t \geqq 2$. By letting

$$\delta = \min\{\min_j \bar{k}_j, \min_i \bar{c}'_i, \min_j \bar{z}'_j, \min_i \hat{c}'_i, \min_j \hat{z}'_j\},$$

we see that these in turn provide the requisite quantity sequences x^t for $t \geqq 0$.

As in the last section, it is again relatively straightforward to establish that $X \subset \ell_1$ is closed, convex and freely disposable. Moreover, again letting the quantity sequence x^* be generated by the efficient growth path under scrutiny, x^* is still a boundary point of X. Hence, the desired conclusion follows directly upon application of the Support Theorem, since (1) implies (4) when $(c_s, z_s, k_s) = (c,z,k) \varepsilon T$ for $s = t$ and $(c_s, z_s, k_s) = (c_s^*, z_s^*, k_s^*)$ for $s \neq t$, and (16) when $k_0 = \bar{k}$, and $(c_t, z_t, k_t) \varepsilon T$ and $k_{t+1} = z_t$ for $t \geqq 0$. $\blacksquare$

It is worth emphasizing here, although we establish a stronger result in the next section, that capital-value transversality also follows from the argument just given. That is, by taking the quantity sequence $x \varepsilon X$ generated by the trivial growth path $(c_t, z_t, k_t) = 0 \varepsilon T$ for $t \geqq 0$, (1) yields

$$\sum_{t=0}^{\infty} p_t \cdot {}^{*}_{t} \geqq q_{-1} \cdot \bar{} .$$

Putting this inequality together with (6), we see immediately that it must be true that

$$\text{(19)} \qquad \lim_{t \to \infty} q_t \cdot k_{t+1}^* = 0.$$

It is also important to be aware of what the Consumption-Value Maximization theorem does not claim: Namely the theorem only

asserts that <u>some</u> associated competitive price system exhibits the property of consumption-value maximization. And it is easy to construct examples where a particular efficient growth path also has an associated competitive price system which does not exhibit this property. The extended example presented in Appendix B is designed in part to illustrate this point. (For a more detailed elaboration in the context of closed, multi-sector models, see the examples and results in [18].) That extended example also illustrates another obvious but important point, that without T6 (or, more generally, without summability in "natural" units of measurement), an efficient growth path may or may not have an associated competitive price system exhibiting consumption-value maximization (even though, in our general model, consumption-value is necessarily bounded).

VI. <u>OPTIMALITY (OR CONSUMPTION-VALUE MAXIMIZATION) AND CAPITAL-VALUE TRANSVERSALITY</u>

Given nonnegative, nontrivial values for consumption goods,

$$\text{(20)} \qquad (\bar{p}_0, \bar{p}_1, \ldots) \geq 0,$$

a particular feasible growth path (denoted once more by asterisks) is <u>optimal</u> if

$$\text{(21)} \qquad \infty > \sum_{t=0}^{\infty} \bar{p}_t \cdot c_t^* \geqq \sum_{t=0}^{\infty} \bar{p}_t \cdot c_t \text{ for every feasible growth path (2).}$$

In the literature on optimal growth theory, it is usually assumed that consumption goods output consists of a single quantity (i.e., that m = 1), which is measured in welfare, or utility terms (refer again to footnote 3), and that consumption good values are constant-rate discount factors (i.e., that $\overline{p}_t = (1+\rho)^{-t}$ for $t \geqq 0$ with $\rho \geqq 0$). Neither specialization is fundamental to the price characterization of optimal growth paths (though both have been central to the detailed description of their evolution).

In this section we use the Support Theorem to develop a short, simple proof of the following important proposition:

Capital-Value Transversality (Peleg and Weitzman): If a particular feasible growth path is optimal, then it has an associated competitive price system (4) such that

$$(22) \qquad p_t = \overline{p}_t \text{ for } t \geqq 0$$

and

$$(19) \qquad \lim_{t\to\infty} q_t \cdot k^*_{t+1} = 0.$$

The converse obtains without qualification.

Proof: For the purposes of this argument, we interpret the quantity sequence x as net final outputs of welfare and investment goods

$$(23) \qquad x = (\sum_{t=0}^{\infty} \overline{p}_t \cdot c_t, \overline{k} - k_0, z_0 - k_1, \ldots, z_t - k_{t+1}, \ldots),$$

the corresponding price sequence π as

$$(24) \qquad \pi = (u, q_{-1}, q_0, \ldots, q_t, \ldots),$$

and the set X as

$$(25) \quad X = \{x : x \in \ell_1, x_0 \leqq \sum_{t=0}^{\infty} \overline{p}_t \cdot c'_t \text{ and } (c'_t, z_t, k_t) \in T \text{ for } t \geqq 0\}.$$

It is easily checked that the quantity sequence

$$(26) \quad x^* = (\sum_{t=0}^{\infty} \overline{p}_t \cdot c^*_t, \overline{k} - k^*_0, z^*_0 - k^*_1, \ldots, z^*_t - k^*_{t+1}, \ldots) = (\sum_{t=0}^{\infty} \overline{p}_t \cdot c^*_t, 0, 0, \ldots)$$

generated by the optimal growth path is a boundary point of X, and has a positive first element $x^*_0 = \sum_{t=0}^{\infty} \overline{p}_t \cdot c^*_t > 0$ (by virtue of the hypothesis $(\overline{p}_0, \overline{p}_1, \ldots) \geq 0$ and maintained assumptions $\overline{k} > 0$ and T5). Also, the set X contains every quantity sequence x generated by a feasible growth path (2), since then $0 \leqq x \leqq x^*$ (by virtue of the definitions (23) and (26)). Finally, as in Section IV, it is once more relatively straightforward to verify that $X \subset \ell_1$ is closed, convex, and freely disposable, but now "productive"[10] rather than productive.

Hence, by the same maneuver performed in Section IV, but now only with respect to the units for measuring net final investment goods outputs (since here consumption goods only appear in terms of welfare), we can easily transform the set X into one which is uniformly "productive" as well (noting that this transformed set still contains every quantity sequence generated by a feasible growth path). Thus, once again utilizing the Support Theorem to derive (1), we can infer:

1. Nontrivial welfare value, i.e., u > 0 or, without loss of generality, u = 1:

Suppose otherwise, that is, $u = 0$. Then $\pi \cdot x^* = 0$, while -- since $\pi_t > 0$ and $x^t \varepsilon X$ for some $t \geqq 0$ -- $\pi \cdot x > 0$ for some $x \varepsilon X$, contradicting (1).

2. Intertemporal welfare-profit maximization, i.e., (4) and (22):

For the quantity sequences generated by the growth paths $(c_s, z_s, k_s) = (c, z, k) \varepsilon T$ for $s = t$ and $(c_s, z_s, k_s) = (c^*_s, z^*_s, k^*_s)$ for $s \neq t$, (1) yields

$$\overline{p}_t \cdot c + q_t \cdot z - q_{t-1} \cdot k_t \leqq \overline{p} \cdot c^*_t + q_t \cdot z^*_t - q_{t-1} \cdot k^*_t = 0$$

for every $(c, z, k) \varepsilon T$ and $t \geqq 0$,

in light of 1 above.

3. Capital-value transversality, i.e., (19):

The argument is identical to that presented toward the end of the last section, in light of 2 above. ■

There is an interesting, if hardly surprising corollary to the Capital-Value Transversality theorem.

Duality: If there exists an optimal solution to the problem

$$\text{(27)} \qquad \text{maximize } \sum_{t=0}^{\infty} \overline{p}_t \cdot c_t \text{ subject to (2),}$$

then there exists an optimal solution to the problem

$$\text{(28)} \qquad \text{minimize } q_{-1} \cdot \overline{k} \text{ subject to (3) and } p_t = \overline{p}_t \text{ for } t \geqq 0,$$

and their optimal values are equal.

Proof: On the one hand, we see from (5) that feasible solutions to (27) and (28) must satisfy

$$\sum_{t=0}^{\infty} \overline{p}_t \cdot c_t \leqq q_{-1} \cdot \overline{k}.$$

On the other hand, appealing to the theorem just proved, we see from (6) that there is a feasible solution to (28) (namely, that associated with the optimal solution to (27)) which satisfies

$$\sum_{t=0}^{\infty} \overline{p}_t \cdot c^*_t = q_{-1} \cdot \overline{k}.$$

But this is then an optimal solution to (28) exhibiting the asserted property.

We note in passing that many of the recent results concerning global stability of optimal growth paths have their ultimate basis in the Capital-Value Transversality theorem (in particular, see [7] and related contributions to the same Symposium).

VII. COMPLICATIONS FROM PRIMARY FACTORS

The general growth model introduced in Section III has one very unrealistic feature; it implicitly rules out the existence of primary factors. In this section we will repair this defect, and in doing so will establish that all but one of the foregoing results remain virtually unaffected.

Suppose now that the feasible growth paths in the economy are described by

$$(2') \quad \begin{cases} (c_t, z_t, k_t, \ell_t) \in T', \ k_{t+1}=z_t \text{ and } \ell_t=\bar{\ell}_t=\bar{\ell}>0 \text{ for } t\geqq 0, \\ k_0=\bar{k}>0 \end{cases}$$

where $\ell = (\ell_0, \ell_1, \ldots, \ell_h)$ is an h-vector of primary factor inputs, $\bar{\ell}$ is an h-vector of their exogenous supply, and $T' = \{(c,z,k,\ell)\}$ is the static technology available for producing outputs from capital stock and primary factor inputs. In fact, there is no real loss of generality in assuming that h = 1, and that $\bar{\ell} = 1$, which we henceforth will.[11] Furthermore, we will assume that T' is again a nonnegative, closed set exhibiting properties analogous to T1-T5:

T'1. T' is convex;

T'2. If $(c,z,k,\ell) \in T'$, $0 \leqq (c',z') \leqq (c,z)$ and $(k',\ell') \geqq (k,\ell)$, then $(c',z',k',\ell') \in T'$;

T'3. T' is a cone;

T'4. If $(c,z,k,\ell) \in T'$ and $(k,\ell) = 0$, then $(c,z) = 0$;

and

T'5. If $(c,z,k,\ell) \in T'$, $\ell > 0$ and $k' > k$, then there exists $(c',z') > (c,z)$ such that $(c',z',k',\ell) \in T'$.

Notice that now T'3 is completely innocuous, since feasible growth paths only involve the projection of a particular cross-section of

T' (namely, the convex set $\{(c,z,k) : (c,z,k,1) \in T')\}$, and any such convex set can be viewed as the projection of a particular cross-section of a convex cone in 1-higher dimension. It is also worth mentioning that T'5 is equivalent -- given our other maintained assumptions (especially T'1-T'3) -- to a seemingly alternative productivity characteristic quite commonly encountered in the capital theory literature since Malinvaud's corrigendum [16]

T'5'. <u>Nontightness in Capital Stocks</u>: If $(c,z,k,\ell) \in T'$ and $\ell > 0$, then there exists $(c'',z'') > (c,z)$, $k'' \geqq 0$ and $0 \leqq \ell'' < \ell$ such that $(c'',z'',k'',\ell'') \in T'$.[12]

(See, for example, the analysis centered around this and other productivity and substitution properties in Kurz [11].)

On the price side of the economy, the description of "feasible" price systems corresponding to (2') is

$$(3') \quad \begin{cases} (p_t, q_t, r_t, w_t) \in M' \text{ and } r_t = q_{t-1} \text{ for } t \geqq 0, \\ q_{-1} \geqq 0 \end{cases}$$

where w is a scalar, say, wage and

$$M'=\{(p,q,r,w):(p,q)\geqq 0 \text{ and } p\cdot c+q\cdot z-r\cdot k-w\cdot \ell \leqq 0 \text{ for every } (c,z,k,\ell)\in T'\}$$

is the nonnegative subset of the dual cone to T'. Hence, an associated competitive price system is now defined as a nontrivial solution to (3') such that

(4') $p_t \cdot c + q_t \cdot z - q_{t-1} \cdot k - w_t \cdot \ell \leq p_t \cdot c_t^* + q_t \cdot z_t^* - q_{t-1} \cdot k_t^* - w_t = 0$

for every $(c,z,k,\ell) \in T'$ and $t \geqq 0$.

Given this amendment to our general growth model, by and large the only modifications needed in order to validate our previous analysis involve accounting for wage payments to the primary factor. Two such modifications are essential:

1. In the first place, in order for the conclusion of the Support Theorem (1) to yield competitive price systems (4'), in each previous application we must reinterpret the quantity sequence x to include net final outputs of primary factors $1-\ell_t$ for $t \geqq 0$ and the price sequence π to include a corresponding wage imputation w_t for $t \geqq 0$.[13] Thus, for example, in the argument establishing Consumption-Value Maximization the quantity sequence (17) becomes

(17') $x = (\bar{k}-k_0, c_0, z_0-k_1, 1-\ell_0, \ldots, c_t, z_t-k_{t+1}, 1-\ell_t, \ldots)$,

while the price sequence (9) becomes

(9') $\pi = (q_{-1}, p_0, q_0, w_0, \ldots, p_t, q_t, w_t, \ldots)$.

2. In the second place, in calculating bounds for consumption-value, we must include an imputation to the stream of primary factors $\bar{\ell}_t = 1$ for $t \geqq 0$ as well as initial capital stocks $\bar{k}$. Thus, for example, the upper bounds (5) and (6) become, respectively,

(5') $\sum_{t=0}^{\mathscr{t}} p_t \cdot c_t \leq \sum_{t=0}^{\mathscr{t}} p_t \cdot c_t + q_{\mathscr{t}} \cdot k_{\mathscr{t}+1} \leq q_{-1} \cdot \bar{k} + \sum_{t=0}^{\mathscr{t}} w_t$ for $\mathscr{t} \geqq 0$

for every feasible growth path (2')

and

(6') $\sum_{t=0}^{\mathscr{t}} p_t \cdot c_t^* \leq \sum_{t=0}^{\mathscr{t}} p_t \cdot c_t^* + q_{\mathscr{t}} \cdot k_{\mathscr{t}+1}^* = q_{-1} \cdot \bar{k} + \sum_{t=0}^{\mathscr{t}} w_t$ for $\mathscr{t} \geqq 0$.[14]

The fact that, in general, we cannot rule out an unbounded imputation

$$q_{-1} \cdot \bar{k} + \sum_{t=0}^{\infty} w_t = q_{-1} \cdot \bar{k} + \lim_{\mathscr{t}\to\infty} \sum_{t=0}^{\mathscr{t}} w_t = \infty$$

has several important consequences, both analytical and conceptual:

a. The corollaries to the Price Characterization theorem (itself modified to include a term $\sum_{t=0}^{\infty} w_t(1-\ell_t)$ on the right-hand side of (11), and a term $\sum_{t=0}^{\infty} |1-\ell_t|$ on the left-hand side of the inequality restricting comparison paths) must be restated in terms of deviations-in-consumption-value $\sum_{t=0}^{\infty} p_t \cdot (c_t - c_t^*)$ (=0 when $c_t = c_t^*$ for $t \geqq 0$), rather than in terms of simple consumption-value $\sum_{t=0}^{\infty} p_t \cdot c_t$ (= $\sum_{t=0}^{\infty} p_t \cdot c_t^*$ when $c_t = c_t^*$ for $t \geqq 0$)

b. The argument at the end of Section V establishing capital-value transversality must be altered to accommodate the fact that, with the reinterpretation (17'), the trivial growth path $(c_t, z_t, k_t, \ell_t) = 0 \in T'$ for $t \geqq 0$ no longer generates a quantity sequence in X (since, by definition, the totality of primary factors is unbounded). This is easily accomplished once we observe that the "almost" trivial growth paths $(c_t, z_t, k_t, \ell_t) = 0 \in T'$ for $0 \leqq t \leqq \mathscr{t}$ and $(c_t, z_t, k_t, \ell_t) = (0,0,0,1) \in T'$ for $\mathscr{t} < t$, for $\mathscr{t} \geqq 0$, do generate quantity sequences in X. Hence, for these growth paths, (1) yields

$$\infty > \sum_{t=0}^{\infty} p_t \cdot c_t^* \geqq q_{-1} \cdot \bar{k} + \sum_{t=0}^{\mathcal{t}} w_t \text{ for } \mathcal{t} \geqq 0$$

or

$$\infty > \sum_{t=0}^{\infty} p_t \cdot c_t^* \geqq q_{-1} \bar{k} + \sum_{t=0}^{\infty} w_t .$$

In conjunction with (6'), this inequality immediately entails capital-value transversality (19).

c. The suitability of consumption-value maximization (at given consumption goods values) as the fundamental notion of optimality must be reappraised, since it may simply be overly restrictive (as it is, for instance, in the canonical one-good model, where the golden rule path does not yield maximum consumption-value at <u>any</u> given consumption goods values).[15] Anything like a complete resolution of this issue is way beyond the scope of our paper -- if not beyond the limit of our ability. However, we will briefly discuss one directly pertinent aspect -- the variety of similar but broader alternatives and their respective price characterizations -- in the following section.

There is a second point in the foregoing analysis at which the introduction of primary factors raises a serious substantive issue. This is in Section V, where we employed T6 to guarantee summability of the whole stream of consumption goods. While it is true that an appropriate analogue,

T'6. There exist $p > 0$ and $q \geqq 0$ such that

$$p \cdot c + q \cdot z - q \cdot k \leqq 0 \text{ for every } (c,z,k,\ell) \in T',$$

also implies

$$(18') \quad \sum_{t=0}^{\infty} \sum_{i=1}^{m} c_{it} < \infty \text{ for every solution to } (2'),$$

this sort of productivity characteristic may now appear -- at least to one steeped in the tradition of neoclassical growth theory -- to be quite restrictive. In particular, since T'6 is easily shown to be equivalent to condition A1 in Appendix A,[16] it automatically excludes the possibility of nontrivial (modified) golden rule paths. But while this may be analytically inconvenient, it is surely not intuitively implausible; the mere availability of primary factors may just not be enough to offset the scarcity of essential exhaustible resources. One precise formulation of this latter intuition is outlined in the last comment in Appendix A.

VIII. <u>ALTERNATIVE NOTIONS OF OPTIMALITY</u>

In order to circumvent the limitations of consumption-value maximization as a procedure for choosing a "best" efficient growth path, a number of related but weaker criteria have been proposed in the literature [4,5,9,29,33]. For our purposes here it is useful to incorporate these in a more systematic listing of candidates, all of which involve evaluating the whole stream of consumption goods at given values (20). In rank order of decreasing strength (i.e., so each entails its successor), one could reasonably judge a particular feasible growth path (denoted, for the last time, by asterisks) to be "optimal" according as to whether it exhibits

1. Consumption-value maximization:

$$\infty > \sum_{t=0}^{\infty} \overline{p}_t \cdot c_t^* \geqq \sum_{t=0}^{\infty} \overline{p}_t \cdot c_t \text{ for every feasible growth path (2'); or}$$

2. Deviations-in-consumption-value maximization (or "best" deviations-in-consumption-value):

$$0 \geqq \lim_{\mathcal{t}\to\infty} \sum_{t=0}^{\mathcal{t}} \overline{p}_t \cdot (c_t - c_t^*) \text{ for every feasible growth path (2'); or}$$

3. "Better" deviations-in-consumption-value:

$$0 \geqq \limsup_{\mathcal{t}\to\infty} \sum_{t=0}^{\mathcal{t}} \overline{p}_t \cdot (c_t - c_t^*) \text{ for every feasible growth path (2'); or}$$

4. "Good" deviations-in-consumption-value:

$$0 \geqq \liminf_{\mathcal{t}\to\infty} \sum_{t=0}^{\mathcal{t}} \overline{p}_t \cdot (c_t - c_t^*) \text{ for every feasible growth path (2'); or}$$

5. Restricted deviations-in-consumption-value maximization:

$$0 \geqq \sum_{t=0}^{\infty} \overline{p}_t \cdot (c_t - c_t^*) \text{ for every feasible growth path (2') such that}$$

$$\sum_{t=0}^{\infty} \sum_{i=1}^{m} u_{it} |c_{it} - c_{it}^*| \leqq \mathcal{B}; \text{ or (only when } \sum_{t=0}^{\infty} \overline{p}_t \cdot c_t^* < \infty)$$

6. Restricted consumption-value maximization:

$$\infty > \sum_{t=0}^{\infty} \overline{p}_t \cdot c_t^* \geqq \sum_{t=0}^{\infty} \overline{p}_t \cdot c_t \text{ for every feasible growth path (2') such that}$$

$$\sum_{t=0}^{\infty} \sum_{i=1}^{m} u_{it} |c_{it} - c_{it}^*| \leqq \mathcal{B},$$

where, in 5 and 6, $u_{it} > 0$ for $1 \leqq i \leqq m$, $t \geqq 0$ are some (fixed) positive units for measuring consumption goods outputs, and $\mathcal{B}$ is some (fixed) finite bound for the totality of their permissible deviations. Since

we know that efficient growth paths are "optimal" in the sense of 5 with primary factors (or in the sense of 6 without, both at associated consumption goods prices), it is quite natural to focus on this criterion in generalizing the Capital-Value Transversality theorem of Section VI; similar results can be obtained in a similar manner for the criteria 2 and 4, but not for 3 (since the lim sup operation does not preserve convexity).

Before we can derive a specific price characterization, however, we will require one additional assumption concerning the possibility of substituting investment goods output for consumption goods output

T'8. Accumulation From Foregone Consumption: If $(c,z,k,\ell) \varepsilon T'$ and $0 \leqq c' \leq c$, then there exists $z' > z$ such that $(c',z',k,\ell) \varepsilon T'$.[17]

Thus armed -- and fortified by preceding applications of the Support Theorem -- it is almost routine to establish the following general result:

Price Characterization of "Optimal" Growth Paths: Suppose T' satisfies T'8. If a particular feasible growth path is "optimal" in the sense of 5, then it has an associated competitive price system (4') such that

$$p_t = \overline{p}_t \text{ for } t \geqq 0 \tag{22}$$

and

$$0 \geqq \sum_{t=0}^{\infty} \overline{p}_t \cdot (c_t - c_t^*) - q_{-1} \cdot (k - \overline{k}) \tag{29}$$

for every growth path which (i) is feasible from nonnegative capital stocks, or satisfies (2') with $\bar{k} = k \geqq 0$, and (ii) yields sufficiently small deviations-in-consumption, or satisfies

$$\sum_{t=0}^{\infty} \sum_{i=1}^{m} u_{it}|c_{it} - c^*_{it}| \leqq B < \infty.$$

The converse obtains without qualification.

Remark: Property (29) is only one way of expressing the fact that the associated capital goods prices reflect their marginal "welfare" value. A more complete catalogue, along with a more leisurely discussion, can be found in [ ].

Proof: We will only sketch the essential differences between this and previous arguments. Here the quantity sequence x is interpreted as net final outputs of "welfare", investment goods and primary factors

$$(30) \quad x = \left(\sum_{t=0}^{\infty} \bar{p}_t \cdot (c_t - c^*_t), \bar{k} - k_0, z_0 - k_1, 1 - \ell_0, \ldots, z_t - k_{t+1}, 1 - \ell_t, \ldots\right),$$

the corresponding price sequence π as

$$(31) \quad \pi = (u, q_{-1}, q_0, w_0, \ldots, q_t, w_t, \ldots),$$

and the set X as further constrained by the by restriction on permissible deviations-in-consumption

$$(32) \quad X = \{x : x \varepsilon \ell_1, x_0 \leqq \sum_{t=0}^{\infty} \bar{p}_t \cdot (c'_t - c^*_t),$$

$$\sum_{t=0}^{\infty} \sum_{i=1}^{m} u_{it}|c'_{it} - c^*_{it}| \leqq B \text{ and } (c'_t, z_t, k_t, \ell_t) \varepsilon T' \text{ for } t \geqq 0\}.$$

Notice that, in particular, the "optimal" growth path generates the quantity sequence $x^* = 0$. Using T'5 and T'8, it is also possible to show that, though X is not uniformly productive, it is productive in the following sense: There exists $x'_0 \leqq 0$ such that for $t \geqq 1$ there exists x^t such that $x^t_0 \geqq x'_0$, $x^t_s > 0$ for $s = t$ and $x^t_s \geqq 0$ for $1 \leqq s \neq t$.[18] So (now skipping a number of steps we have already been through several times before) we can therefore make one last appeal to the Support Theorem to deduce (1). The only critical step left is the verification that, without loss of generality, u = 1. But this follows from the same reasoning we used in the proof of the Capital-Value Transversality theorem in Section VI -- while the rest is basically just a repetition of the proof of the corollary to the Price Characterization theorem in Section IV.

Our interest in investigating this kind of characterization was greatly inspired by the interesting and careful work of Peleg [28] and Peleg-Zilcha [29] relating to the open, multi-sector model. It would be nice, if possible, to elaborate the connection between the seemingly different productivity and substitution properties underlying their and our quite similar results.

APPENDIX A

In this appendix we establish equivalency between T6 and the productivity condition -- defining $\bar{N}$ = closure of N (see equation (15)) --

A1. There is no $(c,y) \in \bar{N}$ such that $c \geq 0$ and $y \geqq 0$,

and sufficiency for T6 of either the substitution condition

A2. <u>Uniform Substitution Between Output of each Consumption Good and Output of all Investment Goods</u>: For every $1 \leqq i \leqq m$ and $\varepsilon_i > 0$ there is a $\delta_i > 0$ such that if $(c,z,k) \in T$ and $c_i \geqq \varepsilon_i$, then $(c',z',k) \in T$ for $c' = c - (0,\ldots, \varepsilon_i,\ldots, 0)$ and $z' = z + \delta_i(1,1,\ldots,1)$,

or the productivity condition -- assuming that capital stock of type 1 is an exhaustible resource (see footnote 5) --

A3. <u>Strict Deterioration of Capital Stocks without Exhaustible Resources</u>: If $(c,z,k) \in T$, $z_1 = k_1 = 0$ and $(c,z,k) \neq 0$, then $z_j < k_j$ for some $2 \leqq j \leqq n$.

<u>Lemma 1A</u>: T6 obtains if and only if A1 does.

<u>Proof</u>: (necessity) Given T6, suppose that A1 were false, i.e., that there is $(c',y') \in \bar{N}$ such that $c' \geq 0$ and $y' \geqq 0$. Then (i) there is a sequence $\{(c^s, y^s)\}$ such that $(c^s, y^s) \in N$ and $\lim_{s\to\infty} (c^s, y^s) = (c', y')$ with a corresponding sequence $\{(c^s, z^s, k^s)\}$ such that $(c^s, z^s, k^s) \in T$ and $y^s = z^s - k^s$, while (ii) $p\cdot c' + q\cdot y' \geqq p\cdot c' > 0$. Hence, for sufficiently large s,

$$(c^s,z^s,k^s) \in T \text{ but } p\cdot c^s + q\cdot z^s - q\cdot k^s = p\cdot c^s + q\cdot y^s > 0,$$

contradicting T6.

(sufficiency) We will use the easily verified fact that, given our maintained assumptions on T, $\bar{N}$ is a closed, convex cone exhibiting free disposal in y (i.e., if $(c,y) \in \bar{N}$ and $y' \leqq y$, then $(c,y') \in \bar{N}$).

Given A1, let $0 \leqq n' \leqq n$ be such that if $1 \leqq j' \leqq n'$ $(n' + 1 \leqq j' \leqq n)$, then there is no (some) $(c,y) \in \bar{N}$ such that $c = 0$ and $y_j > 0$ for $j = j'$, $y_j = 0$ for $j \neq j'$. Partitioning $y = (y^1,y^2)$ accordingly, consider the set

$$U = \{(c,y^1): (c,y) \in \bar{N} \text{ and } y^2 = 0\}.$$

Since $\bar{N}$ is a closed, convex cone, so is U. Moreover, the hypothesis A1 together with the choice of n' imply that there is no $(c,y^1) \in U$ such that $(c,y^1) \geq 0$. Hence, appealing to a duality result characterizing 0 as a maximal point of a closed, convex cone (see Nikaido [21], pp. 35-36), we know that there exists $(p,q^1) > 0$ such that

$$\text{A(1)} \qquad (p,q^1)\cdot(c,y^1) \leqq 0 \text{ for every } (c,y^1) \in U.$$

Now consider the set

$$V = \{(x,y^2): x \leqq (p,q^1)\cdot(c,y^1) \text{ and } (c,y) \in \bar{N}\}.$$

Since $\bar{N}$ is a convex cone exhibiting free disposal in y, V is a convex cone exhibiting free disposal (i.e., if $(x,y^2) \in V$ and $(x',y^{2\prime}) \leqq (x,y^2)$, then $(x',y^{2\prime}) \in V$). Moreover, in light of A(1), 0 must be a boundary point of V. Hence, using a standard support theorem (see again, for instance, Nikaido [21] p. 35), we know that there exists $(\lambda,q^2) \geq 0$ such that

$$\lambda\cdot x + q^2\cdot y^2 \leqq 0 \text{ for every } (x,y^2) \in V$$

or

A(2) $\lambda(p,q^1)\cdot(c,y^1) + q^2\cdot y^2 \leqq 0$ for every $(c,y) \in \bar{N}$.

But without loss of generality $\lambda = 1$, since if $\lambda = 0$, then $q^2 \geq 0$ and, again by the choice of n', there is some $(0,(0,y^2)) \in \bar{N}$ such that $q^2\cdot y^2 > 0$, contradicting A(2). Hence, we have established that there exist $p > 0$ and $q = (q^1,q^2) \geqq 0$ such that

$$p\cdot c + q\cdot y \leqq 0 \text{ for every } (c,y) \in \bar{N},$$

which implies the desired conclusion

$$p\cdot c + q\cdot z - q\cdot k \leqq 0 \text{ for every } (c,z,k) \in T$$

(since $N \subset \bar{N}$). $\blacksquare$

It is worth mentioning explicitly that T5 requires that $q \geq 0$ in T6 (so that, for example, our use of T6 in the text makes sense).

Lemma 2A: If A2 obtains, then so does T6.

Proof: For this argument we require the result that the units for measuring investment goods output can be specified -- consistent with our maintained assumptions together with T7 -- so that the following productivity limitation is satisfied:

A4. Impossibility of Sustained Capital Accumulation: There is no $(c,y) \in \bar{N}$ such that $c \geqq 0$ and $y > 0$.

Since this result is established in exactly the same fashion as the well-known result that the von Neumann growth rate in the closed, multi-sector model[19] can be specified equal to zero (see, in particular, the discussion and references

in [14]), we omit the details of its proof.

What we will show explicitly, then, is that A2 and A4 imply A1 or, by virtue of Lemma 1A, T6: Given A2 and A4, suppose that A1 were false i.e., that there is some $(c',y') \in \bar{N}$ such that $c' \geq 0$ and $y' \geqq 0$ (for definiteness assume $c'_i > 0$ for $i = i'$, $c'_i \geqq 0$ for $i \neq i'$), and hence, that there is a sequence $\{(c^s,y^s)\}$ such that $(c^s,y^s) \in N$ and $\lim_{s\to\infty} (c^s,y^s) = (c',y')$ with a corresponding sequence $\{(c^s,z^s,k^s)\}$ such that $(c^s,z^s,k^s) \in T$ and $y^s = z^s - k^s$. Then, there must be some $\varepsilon_{i'} > 0$ such that, for sufficiently large s, $c^s_{i'} \geqq \varepsilon_{i'}$, and hence, by virtue of A2, some $\delta_{i'} > 0$ such that, again for sufficiently large s, $(c^{s'},z^{s'},k^s) \in T$ or $(c^{s'},y^{s'}) \in N$ for

$$c^{s'} = c^s - (0,\ldots,\varepsilon_{i'},\ldots 0),\ z^{s'} = z^s + \delta_{i'}(1,1,\ldots,1) \text{ and}$$

$$y^{s'} = y^s + \delta_{i'}(1,1,\ldots,1).$$

But this means that $\lim_{s\to\infty} (c^{s'},y^{s'}) = (c' - (0,\ldots,\varepsilon_{i'},\ldots,0),\ y' + \delta_{i'}(1,1,\ldots,1)) \in \bar{N}$, contradicting A4. $\blacksquare$

Lemma 3A: If A3 obtains, then so does T6.

Proof: What we will show here is that, when capital stock of type 1 is an exhaustible resource, A3 implies A1, and hence T6: Given A3, once again suppose that A1 were false, i.e., that ... (as in the preceding proof)... . Because capital stock of type 1 is an exhaustible resource, we can also suppose that $z_1^s = 0$ and $y_1^s = -k_1^s$ or $\lim_{s\to\infty} z_1^s = \lim_{s\to\infty} k_1^s = 0$ (since $\lim_{s\to\infty} (z_1^s - k_1^s) = \lim_{s\to\infty} y_1^s = y'_1 \geqq 0$). Now consider the sequence $\{x^s\}$ such that

$$x^s = (c^s,z^s,k^s)/||(c^s,z^s,k^s)|| \in T$$

(since T is homogeneous). By the Bolzano-Weierstrass Theorem this sequence must have an accumulation point, say, $x = (c,z,k)$. But then $(c,z,k) \in T$ (since T is closed), $z_1 = k_1 = 0$ (since $\lim_{s\to\infty} z_1^s = \lim_{s\to\infty} k_1^s = 0$ but $\lim_{s\to\infty} c^s = c' \geq 0$) and $(c,z,k) \geq 0$ (since $||x^s|| = 1$), but $z_j \geqq k_j$ for all $2 \leqq j \leqq n$ (since $\lim_{s\to\infty} (z_j^s - k_j^s) = \lim_{s\to\infty} y_j^s = y_j' \geqq 0$), contradicting A3.

Several comments are pertinent to the foregoing results:

1. If T is polyhedral, then N is polyhedral, and $\overline{N} = N$ (see, for instance, Rockafellar [31] pp. 171 and 175). In this case (or, more generally, whenever $\overline{N} = N$), because we need only consider single points in N and T rather than sequences of points in N and T, Lemma 1A immediately reveals that T6 is necessary as well as sufficient for summability. Moreover, for the same reason, Lemmas 2A and 3A remain true with some weakening of A2 and A3, respectively. In particular, we can relax the uniformity requirement in A2 (to read "For every $1 \leqq i \leqq m$ and $\varepsilon_i > 0$ if $(c,z,k) \in T$ and $c_i \geqq \varepsilon_i$, then there is a $\delta_i > 0$ such that $(c',z',k) \in T$ ..." -- so that δ_i can depend on (c,z,k) rather than just c_i), and the strictness requirement in A3 (to read "... then $c = 0$ or $z_j < k_j$ for some $2 \leqq j \leqq n$."-- so that $(0,k,k) \in T$ such that $k_j > 0$ for some $2 \leqq j \leqq n$ is permitted; see comment 4 below).

2. We can directly generalize A3 and Lemma 3A when capital stocks of type $1,2,\ldots,n' \leqq n$ are exhaustible resources (by simply replacing z_1,k_1 and 2 with $z^1 = (z_1,z_2,\ldots,z_{n'})$, $k^1 = (k_1,k_2,\ldots,k_{n'})$ and n', respectively). This sort of productivity condition seems to us a very natural way of modelling the most pessimistic (as well as most unimaginative) "limits of growth" prospect.

3. Neither A2 nor A3 is necessary for T6. This will be demonstrated shortly, in terms of the example we examine in the following appendix.

4. There are alternative substitution conditions which entail T6 when T7 is strengthened to

T7'. Possibility of Storage without Sustenance: There exists $0 < \hat{\hat{k}} \leqq \overline{k}$ such that $(0,\hat{\hat{k}},\hat{k}) \in T$.

For discussion and analysis of some such conditions in the context of the closed, multi-sector model we once more refer the interested reader to [14]. Note, however, that while A3 is consistent with T7, it is not consistent with T7'.

5. When there are primary factors in the economy, analogues of neither A2 nor A3 are sufficient for T'6. However, if in addition to

A'3. If $(c,z,k,\ell) \in T'$, $z_1 = k_1 = 0$ and $k \neq 0$, then $z_j < k_j$ for some $2 \leqq j \leqq n$,

we have

A'5. Limited Productivity of Primary Factors: If $(c,z,k,\ell) \in T'$ and $|\ell' \geqq ||k||$, then $(c,z,k,\ell') \in T'$,

then minor changes in the argument establishing Lemma 3A yield.

Lemma 3'A: If A'3 and A'5 obtain, then so does T'6.

Proof: What we will show is that, when capital stock of type 1 is an exhaustible resource, A'3 and A'5 imply A1, and hence T'6: Given A'3 and A'5, suppose that A1 were false, i.e., ... (as in the proof of Lemma 3A but with (c^s, z^s, k^s, ℓ^s) and T' substituted for (c^s, z^s, k^s) and T, respectively) ... . As before we can also suppose that $\lim_{s\to\infty} z_1^s = \lim_{s\to\infty} k_1^s = 0$, while here A'5 enables us to suppose that $\ell^s = \| k^s \|$. Now consider

$$x^s = (c^s,z^s,k^s,\ell^s)/\| (c^s,z^s,k^s,\ell^s)\| \in T' \text{ for } s \geqq 0$$

with accumulation point $x = (c,z,k,\ell)$. Then it must be true that $(c,z,k,\ell) \varepsilon T'$, $z_1 = k_1' = 0$ and $k \neq 0$ (since $\| x \| = 1$ while if $k = 0$, then by virtue of A'5 $\ell = 0$, then by virtue of T'4 $(c,z) = 0$), but $z_j \geqq k_j$ for all $2 \leqq j \leqq n$, contradicting A'3. ∎

Note finally that A'3 and A'5 are consistent with the analogue of T7

T'7. There exist $0 \leq \hat{k} \leqq \bar{k}$ and $\hat{z} > 0$ such that both $(0,\hat{k},\hat{k},1) \varepsilon T'$ and $(0,\hat{z},\hat{k},1) \varepsilon T'$.

APPENDIX B

In this appendix we elaborate an extended example of our general model, one in which there are only two types of both consumption goods and capital stocks. Capital stock of type 1 is an exhaustible resource, while capital stock of type 2 serves as both an inventory of consumption goods and -- in a process using up the exhaustible resource as raw material -- an originator of consumption goods. Besides generally illustrating the Consumption-Value Maximization theorem of Section V, the example specifically demonstrates that

1. Neither A2 nor A3 is necessary for T6;
2. The question of whether T6 is necessary for summability (i.e., $\sum_{t=0}^{\infty} \sum_{i=1}^{m} c_{it} < \infty$ for every solution to (2)) is both open and subtle;
3. In fact, summability is not necessary for consumption-value maximization; and
4. Even with summability, there may be some associated competitive price system which does not exhibit consumption-value maximization, while without summability, there may be no associated competitive price system which exhibits consumption-value maximization.

1. <u>Example</u>

$m = n = 2$

$T = \{(c,z,k) : c_1 \leqq h(k_1 - z_1, k_2),\ z_1 \leqq k_1, c_2 + z_2 \leqq k_2 \text{ and } (c,z,k) \geqq 0\}$

where

$h(x_1,x_2)$ for $(x_1,x_2) \geqq 0$ is differentiable, nonnegative (strictly positive for $(x_1,x_2) > 0$), increasing (strictly increasing for $(x_1,x_2) > 0$), concave and linear homogeneous, and satisfies

$h(0,x_2) = h(x_1,0) = 0$

and

<u>Case 1</u>: $\lim_{x_1 \to 0^+} h_1(x_1,x_2) < \infty$ for $x_2 \geqq 0$

e.g., $h(x_1,x_2) = (1 - e^{-(x_1/x_2)})x_2$

or

<u>Case 2a</u>: $\lim_{x_1 \to 0^+} h_1(x_1,x_2) = \infty$ for $x_2 > 0$

and

for every $0 < x_1' < \infty$, $x_2' > 0$ there is some $0 < \alpha < 1$ such that

$$\frac{h_1(x_1,x_2)x_1}{h(x_1,x_2)} \geqq \alpha \text{ for } 0 \leqq x_1 \leqq x_1',\ x_2 \geqq x_2'$$

e.g., $h(x_1,x_2) = x_1^{\alpha} x_2^{1-\alpha}$ with $0 < \alpha < 1$

or

<u>Case 2b</u>: $\lim_{x_1 \to 0^+} h_1(x_1,x_2) = \infty$ for $x_2 > 0$

and

for some $0 < x_1' < \infty$, $x_2' > 0$ there is no $0 < \alpha < 1$ such that

$$\frac{h_1(x_1,x_2)x_1}{h(x_1,x_2)} \geqq \alpha \text{ for } 0 \leqq x_1 \leqq x_1',\ x_2 \geqq x_2'$$

$$\text{e.g., } h(x_1,x_2) = \begin{cases} (-\log(x_1/x_2))^{-\beta}x_2 & \text{for } 0 \leqq x_1/x_2 \leqq e^{-\beta-1} \\ \tilde{h}(x_1/x_2)x_2 & \text{otherwise} \end{cases}$$

with $\beta > 0$, $\tilde{h}(x)$ for $x \geqq e^{-\beta-1}$ any differentiable, strictly increasing and concave function such that $\tilde{h}(e^{-\beta-1}) = (\beta+1)^{-\beta}$ and $\tilde{h}'(e^{-\beta-1}) = (\beta+1)^{-\beta-1}\beta e^{\beta+1}$.

One can easily verify that this particular technology satisfies all of our maintained assumptions. It also satisfies the strong version of T7 mentioned at the end of the preceding appendix, T7'. (Indeed, it is obvious that the largest possible growth rate of both capital stocks -- achieved only when there is no consumption goods output -- is zero.) Notice finally that this technology does not satisfy either A2 (since, for example, consumption good of type 2 cannot be converted into investment good of type 1) or A3 (since capital stock of type 2 can be costlessly stored); shortly, we will show that it satisfies T6 only in Case 1.

2. <u>Results</u>

The example was chosen in large part because it permits direct characterization of efficiency. This in turn permits full concentration on the property of consumption-value maximization.

a. <u>Efficient Growth Paths</u>: A feasible growth path is efficient if and only if <u>outputs are not obviously wasted</u>, i.e., consumption goods outputs are never freely disposed, $c_{1t} = h(k_{1t} - z_{1t}, k_{2t})$ and $c_{2t} = k_{2t} - z_{2t}$ for $t \geqq 0$, and <u>capital stocks are not superfluously maintained</u>, i.e., resources are eventually exhausted, $\lim_{t\to\infty} k_{1t} = 0$, while (i) if resources are exhausted in finite time, then inventories are eventually depleted, if there is a $\bar{t}_1 < \infty$ such that $k_{1t} > 0$ for $0 \leqq t \leqq \bar{t}_1$ and $k_{1t} = 0$ for $\bar{t}_1 < t$, then $\lim_{t\to\infty} k_{2t} = 0$, and (ii) if inventories are depleted in finite time, then resources are exhausted by the same time, if there is a $\bar{t}_2 < \infty$ such that $k_{2t} > 0$ for $0 \leqq t \leqq \bar{t}_2$ and $k_{2t} = 0$ for $\bar{t}_2 < t$, then $k_{1t} = 0$ for $\bar{t}_2 < t$ (or, alternatively, if $k_{1t} > 0$, then $k_{2t} > 0$ for $t \geqq 0$).

Proof: For simplicity, here we will denote the given path without superscript, and various comparison paths with prime superscript. (necessity)(i) No obvious waste: Suppose otherwise, i.e., $c_{1t} < h(k_{1t} - z_{1t}, k_{2t})$ or $c_{2t} < k_{2t} - z_{2t}$ for some t. Then, $c'_{1t} = h(k_{1t} - z_{1t}, k_{2t}) > c_{1t}$ or $c'_{2t} = k_{2t} - z_{2t} > c_{2t}$ is feasible, ceteris paribus, contradicting the hypothesis. (ii) No superfluous maintenance: Note first that, since $0 \leqq k_{t+1} \leqq k_t$ for $t \geqq 0$, we know $\lim_{t\to\infty} k_t = k_\infty \geqq 0$. Then, on the one hand, suppose $\lim_{t\to\infty} k_{1t} = k_{1\infty} > 0$. It follows that $c'_{10} = h(k_{10} - z_{10} + k_{1\infty}, k_{20}) > c_{10}$, $k'_{1t} = k_{1t} - k_{1\infty} \geqq 0$ for $t > 0$ is feasible, ceteris paribus, contradicting the hypothesis. On the other hand, suppose first that $k_{1t} > 0$ for $0 \leqq t \leqq \bar{t}_1 < \infty$, $k_{1t} = 0$ for $\bar{t}_1 < t$ and $\lim_{t\to\infty} k_{2t} = k_{2\infty} > 0$. Then $c'_{2\bar{t}_1} = k_{2\bar{t}_1} - z_{2\bar{t}_1} + k_{2\infty} > c_{2\bar{t}_1}$, $k'_{2t} = k_{2t} - k_{2\infty} \geqq 0$ for $\bar{t}_1 < t$ is feasible, ceteris paribus, again contradicting the hypothesis. Now, suppose second that $k_{2t} > 0$ for $0 \leqq t \leqq \bar{t}_2 < \infty$, $k_{2t} = 0$ for $\bar{t}_2 < t$ and $k_{1\bar{t}_2+1} > 0$. Then (since $c_{1t} = h(k_{1t} - z_{1t}, 0) = 0$ for $\bar{t}_2 < t$), $c'_{10} = h(k_{10} - z_{10} + k_{1\bar{t}_2+1}, k_{20}) > c_{10}$,

$k'_{1t} = 0$ for $\bar{t}_2 < t$ is feasible, ceteris paribus, also contradicting the hypothesis.

(sufficiency) We begin by noting that when there is no obvious waste, feasible capital stocks obey the dynamical equations

B(1) $$k_{1t+1} = k_{10} + \sum_{s=0}^{t} (k_{1s+1} - k_{1s}) = \bar{k}_1 - \sum_{s=0}^{t} (k_{1s} - z_{1s}) \text{ for } t \geqq 0$$

and

B(2) $$k_{2t+1} = k_{20} + \sum_{s=0}^{t} (k_{2s+1} - k_{2s}) = \bar{k}_2 - \sum_{s=0}^{t} (k_{2s} - z_{2s}) = \bar{k}_2 - \sum_{s=0}^{t} c_{2s} \text{ for } t \geqq 0.$$

We then proceed by considering a number of mutually exclusive and completely exhaustive subcases. For each subcase the logic of the argument is the same; namely, we show that the supposition of a dominating (refer to equation (7)) feasible growth path leads to a contradiction:

(i) Suppose first that $k_{1t} > 0$ for $t \geqq 0$ and $\lim_{t\to\infty} k_{1t} = 0$. (a) If $c'_t \geqq c_t$ for $t \geqq 0$ and there is some $t_2 < \infty$ such that $c'_{2t_2} > c_{2t_2}$, then from B(2)

$$k'_{2t+1} = \bar{k}_2 - \sum_{s=0}^{t} c'_{2s} =$$

$$\bar{k}_2 - \sum_{s=0}^{t} c_{2s} - \sum_{s=0}^{t} (c'_{2s} - c_{2s}) \leqq \begin{cases} k_{2t+1} & \text{for } 0 \leqq t < t_2 \\ k_{2t+1} - (c'_{2t_2} - c_{2t_2}) < k_{2t+1} & \text{for } t_2 \leqq t \end{cases}$$

But by the last stated hypothesis of the proposition, we know that $k_{2t} > 0$ for $t \geqq 0$. Hence, $c_{1t} > 0$ for some $t > t_2$ or, without loss of generality, $c_{1t_2+1} > 0$ or

$$k'_{1t}-z'_{1t} \begin{Bmatrix} \geqq \\ \\ > \end{Bmatrix} k_{1t}-z_{1t} \quad \text{as} \quad t \begin{Bmatrix} \neq \\ \\ = \end{Bmatrix} t_2+1$$

(since $c'_{1t} = h(k'_{1t}-z'_{1t},k'_{2t}) \begin{Bmatrix} \geqq \\ \\ > \end{Bmatrix} h(k_{1t}-z_{1t},k_{2t}) = c_{1t}$ as $t \begin{Bmatrix} \neq \\ \\ = \end{Bmatrix} t_2+1$

and h is increasing in x) or from B(1)

$$k'_{1t+1}=\bar{k}_1-\sum_{s=0}^{t}(k'_{1s}-z'_{1s})=\bar{k}_1-\sum_{s=0}^{t}(k_{1s}-z_{1s})-\sum_{s=0}^{t}[(k'_{1s}-z'_{1s})-(k_{1s}-z_{1s})] \leqq$$

$$\begin{cases} k_{1t+1} & \text{for } 0 \leqq t \leqq t_2 \\ k_{1t+1}-[(k'_{1t_2+1}-z'_{1t_2+1})-(k_{1t_2+1}-z_{1t_2+1})] & \text{for } t_2 < t \end{cases}$$

or

$$\lim_{t\to\infty} k'_{1t} \leqq -[(k'_{1t_2+1}-z'_{1t_2+1})-(k_{1t_2+1}-z_{1t_2+1}) < 0,$$

which is infeasible. (b) If (ruling out the subcase already considered) $c'_{1t} \geqq c_{1t}$ and $c'_{2t} = c_{2t}$ for $t \geqq 0$ and there is some $t_1 < \infty$ such that $c'_{1t_1} > c_{1t_1}$, then now from B(2)

$$k'_{2t+1}=\bar{k}_2-\sum_{s=0}^{t}c'_{2s}=\bar{k}_2-\sum_{s=0}^{t}c_{2s} = k_{2t+1} \text{ for } t \geqq 0$$

or

$$k'_{1t}-z'_{1t} \begin{Bmatrix} \geqq \\ \\ > \end{Bmatrix} k_{1t}-z_{1t} \text{ as } t \begin{Bmatrix} \neq \\ \\ = \end{Bmatrix} t_1$$

(since here $c'_{1t}=h(k'_{1t}-z'_{1t},k'_{2t})\begin{Bmatrix} \geqq \\ \\ > \end{Bmatrix}h(k_{1t}-z_{1t},k_{2t})=c_{1t}$ as $t\begin{Bmatrix} \neq \\ \\ = \end{Bmatrix}t_1$

and h is increasing in x) or again from B(1)

$$k'_{1t+1}=\bar{k}_1-\sum_{s=0}^{t}(k'_{1s}-z'_{1s})=\bar{k}_1-\sum_{s=0}^{t}(k_{1s}-z_{1s})-\sum_{s=0}^{t}[(k'_{1s}-z'_{1s})-(k_{1s}-z_{1s})] \leqq$$

$$\begin{cases} k_{1t+1} & \text{for } 0 \leqq t < t_1 \\ k_{1t+1}-[(k'_{1t_1}-z'_{1t_1})-(k_{1t_1}-z_{1t_1})] & \text{for } t_1 \leqq t \end{cases}$$

or

$$\lim_{t\to\infty} k'_{1t} \leqq -[(k'_{1t_1}-z'_{1t_1})-(k_{1t_1}-z_{1t_1})] < 0,$$

which is also infeasible.

(ii) Suppose second that $k_{1t} > 0$ for $0 \leqq t \leqq \bar{t}_1, < \infty$, $k_{1t} = 0$ for $\bar{t}_1 < t$ and $\lim_{t\to\infty} k_{2t} = 0$. (a) If $c'_t \geqq c_t$ for $t \geqq 0$ and there is some $t_2 < \infty$ such that $c'_{2t_2} > c_{2t_2}$, then (here and after deleting obvious repetitions of parts of the preceding argument)

$$k'_{2t+1} = \ldots \leqq \begin{cases} k_{2t+1} & \text{for } 0 \leqq t < t_2 \\ k_{2t+1} - (c'_{2t_2} - c_{2t_2}) & \text{for } t_2 \leqq t \end{cases}$$

or

$$\lim_{t\to\infty} k'_{2t} \leqq -(c'_{2t_2} - c_{2t_2}) < 0,$$

which is infeasible. (b) If (ruling out the subcase already considered) $c'_{1t} \geqq c_{1t}$ and $c'_{2t} = c_{2t}$ for $t \geqq 0$ and there is some $t_1 < \infty$

such that $c'_{1t_1} > c_{1t_1}$, then from B(2) $k'_{2t+1} = k_{2t+1}$ for $t \geqq 0$ or

$$k'_{1t} - z'_{1t} \left\{\begin{matrix}\geqq\\ >\end{matrix}\right\} k_{1t} - z_{1t} \text{ as } t \left\{\begin{matrix}\neq\\ =\end{matrix}\right\} t_1$$

or from B(1)

$$k'_{1t+1} = \dots \leqq \begin{cases} k_{1t+1} & \text{for } 0 \leqq t < t_1 \\ k_{1t+1} - [(k'_{1t_1} - z'_{1t_1}) - (k_{1t_1} - z_{1t_1})] & \text{for } t_1 \leqq t \end{cases}$$

or

$$k'_{1t+1} \leqq - [(k'_{1t_1} - z'_{1t_1}) - (k_{1t_1} - z_{1t_1})] < 0 \text{ for } \max\,[\bar{t}_1, t_1] \leqq t,$$

which is, once again, infeasible.

b. <u>Conditions for Summability</u>: Though in all three cases, actual capital accumulation is infeasible, and only eventual resource exhaustion is efficient, in Case 1 condition T6 obtains, so that

B(3) $$\sum_{t=0}^{\infty} (c_{1t} + c_{2t}) \leqq b(\bar{k}) < \infty \text{ for every solution to (2),}$$

while in Case 2 (i.e., both Cases 2a and 2b) condition T6 does not obtain, in fact

B(4) $$\sum_{t=0}^{\infty} (c_{1t} + c_{2t}) = \infty \text{ for some solution to (2).}$$

<u>Proof</u>: (Case 1) All that needs establishing is that there is some $p > 0$ and $q \geqq 0$ such that

$$p \cdot c + q \cdot z - q \cdot k \leqq 0 \text{ for every } (c,z,k) \in T.$$

But any stationary competitive price system of the form

B(5) $p_1 = 1$, $0 < h_1(0,1) \leqq q_1 < \infty$ and $p_2 = q_2 > 0$

will do, since if (p,q) satisfies B(5) and $(c,z,k) \in T$, then (using various properties of h)

$$p \cdot c + q \cdot z - q \cdot k \leqq h(k_1 - z_1, k_2) - q_1(k_1 - z_1) - q_2(k_2 - z_2 - c_2) =$$

$$\begin{cases} -q_1(k_1 - z_1) - q_2(k_2 - c_2 - z_2) \leqq 0 & \text{when } k_2 = 0 \\ [h(^{(k_1 - z_1)}/_{k_2}, 1) - q_2(^{(k_1 - z_1)}/_{k_2})]k_2 - q_2(k_2 - z_2 - c_2) \leqq & \\ (h_1(0,1) - q_1)(k_1 - z_1) - q_2(k_2 - c_2 - z_2) \leqq 0 & \text{when } k_2 > 0. \end{cases}$$

From the argument detailed earlier in Section V, we can even calculate a minimum value for the bound in B(3):

$$b(\bar{k}) = h_1(0,1)\bar{k}_1 + \bar{k}_2 = \min_{0 \leqq q_2 \leqq 1} (h_1(0,1)/q_2)\bar{k}_1 + \bar{k}_2 \leqq$$

$$(h,(0,1)\bar{k}_1 + q_2\bar{k}_2)/\min\{1, q_2\} \leqq q \cdot \bar{k}/\min\{p_1, p_2\}.$$

It is worth emphasizing once more that in this case the technology satisfies T6 without satisfying either A2 or A3.

(Case 2) All that needs establishing is B(4) (since we demonstrated earlier that T6 implies B(3), or equivalently, that B(4) implies the denial of T6). The argument involves constructing a feasible growth path which yields

$$\text{B(6)}\qquad \sum_{t=0}^{\infty}(c_{1t}+c_{2t}) = \sum_{t=0}^{\infty} c_{1t} = \infty.$$

In particular, consider the efficient growth path with capital stocks described by

$$k_{1t} = \bar{k}_1 - \sum_{s=0}^{t-1}\varepsilon_s \text{ and } k_{2t} = \bar{k}_2 \text{ for } t \geqq 0,$$

where $\varepsilon_t = k_{1t} - z_{1t} > 0$ for $t \geqq 0$ with $\sum_{t=0}^{\infty}\varepsilon_s = \bar{k}_1$ is chosen as follows: Pick constants $0 < \alpha < 1$ and $\beta > 0$ and let the subsequence of periods $\{t_s\}$ be such that

$$t_0 = 0$$

and

$$t_{s+1} = \min\left\{t : t > t_s \text{ and } h_1\left(\frac{(1-\alpha)\bar{k}_1\alpha^s}{t-t_s}, \bar{k}_2\right)(1-\alpha)\bar{k}_1\alpha^s \geqq \beta\right\} \text{ for } s \geqq 0.$$

(The last is legitimate since Case 2 is characterized by the property that $\lim_{x_1\to 0^+} h_1(x_1,\bar{k}_2) = \infty$ for $\bar{k}_2 > 0$.) Finally, define

$$\varepsilon_t = \frac{(1-\alpha)\bar{k}_1\alpha^s}{t_{s+1}-t_s} > 0 \text{ for } t_s \leqq t < t_{s+1},\ s \geqq 0,$$

which entails

$$0 < \sum_{t=0}^{\mathcal{t}}\varepsilon_t < \sum_{t=0}^{\infty}\varepsilon_t = \sum_{s=0}^{\infty}\ \sum_{u=t_s}^{t_{s+1}-1}\frac{(1-\alpha)\bar{k}_1\alpha^s}{t_{s+1}-t_s} = (1-\alpha)\bar{k}_1\sum_{s=0}^{\infty}\alpha^s = \bar{k}_1 \text{ for } \mathcal{t} \geqq 0.$$

Along this particular feasible growth path consumption goods are produced at the rates

$$c_{1t} = h(\varepsilon_t,\bar{k}_2) \geqq h_1(\varepsilon_t,\bar{k}_2)\varepsilon_t \text{ and } c_{2t} = 0 \text{ for } t \geqq 0$$

(since $0 = h(0,\bar{k}_2) \leqq h(\varepsilon,\bar{k}_2) + h_1(\varepsilon,\bar{k}_2)(0-\varepsilon)$ or $h(\varepsilon,\bar{k}_2) \geqq h_1(\varepsilon,\bar{k}_2)\,\varepsilon$ for $\varepsilon > 0$ by concavity of h). Hence, it follows that

$$\sum_{t=0}^{t_{\delta+1}-1}(c_{1t}+c_{2t}) = \sum_{t=0}^{t_{\delta+1}-1} c_{1t} \geqq \sum_{t=0}^{t_{\delta+1}-1} h_1(\varepsilon_t,\bar{k}_2)\varepsilon_t = \sum_{s=0}^{\delta}\ \sum_{u=t_s}^{t_{s+1}-1} h_1(\varepsilon_{t_s},\bar{k}_2)\varepsilon_{t_s} =$$

$$\sum_{s=0}^{\delta}\ \sum_{u=t_s}^{t_{s+1}-1} h_1\left(\frac{(1-\alpha)\bar{k}_1\alpha^s}{t_{s+1}-t_s},\bar{k}_2\right)\left(\frac{(1-\alpha)\bar{k}_1\alpha^s}{t_{s+1}-t_s}\right) =$$

$$\sum_{s=0}^{\delta} h_1\left(\frac{(1-\alpha)\bar{k}_1\alpha^s}{t_{s+1}-t_s},\bar{k}_2\right)\sum_{u=t_s}^{t_{s+1}-1}\frac{(1-\alpha)\bar{k}_1\alpha^s}{t_{s+1}-t_s} =$$

$$\sum_{s=0}^{\delta} h_1\left(\frac{(1-\alpha)\bar{k}_1\alpha^s}{t_{s+1}-t_s},\ \bar{k}_2\right)(1-\alpha)\bar{k}_1\alpha^s \geqq \sum_{s=0}^{\delta}\beta = (\delta+1)\,\beta \text{ for } \delta \geqq 0,$$

from which we immediately deduce B(6).

We should mention explicitly that the foregoing analysis has shown that, in this example anyway, T6 is necessary as well as sufficient for summability. We have also observed the same result in other easily calculable examples -- for all of which $\bar{N} \neq N$ (see comment 1 at the end of Appendix A). Whether this is just a peculiarity of relatively simple specializations of our general model remains to be seen.

c. Consumption-Value Maximization: (For simplicity, here too we will denote a given efficient growth path without superscript, and various comparison growth paths with prime superscript.) Given an efficient growth path, in Cases 1 and 2a (i) there is some associated competitive price system such that

$$B(7)\qquad \lim_{t\to\infty} q_t \cdot k_{t+1} = 0,$$

and hence (referring to the discussion at the end of Section III)

$$B(8)\qquad \infty > \sum_{t=0}^{\infty} p_t \cdot c_t \geqq \sum_{t=0}^{\infty} p_t \cdot c_t' \qquad \text{for every (primed) solution to (2),}$$

but (ii) there may also be some associated competitive price system such that neither statement is true, while in Case 2b there may simply be no associated competitive price system such that either statement is true.

Proof: We begin by noting that the "pure inventory" competitive price system

$$B(9)\qquad p_{1t} = q_{1t-1} = 0 \text{ and } p_{2t} \begin{Bmatrix} \leq \\ = \end{Bmatrix} q_{2t-1} = 1 \text{ as } c_{2t} \begin{Bmatrix} = \\ > \end{Bmatrix} 0 \text{ for } t \geqq 0$$

is associated with every efficient growth path, since if prices satisfy B(9) and quantities satisfy

$$c_{2t} = k_{2t} - z_{2t} \text{ and } (c_{2t}, z_{2t}, k_{2t}) \geqq 0 \text{ for } t \geqq 0,$$

then

$$p_t \cdot c + q_t \cdot z - q_{t-1} \cdot k = p_{2t} c_2 + q_{2t} z_2 - q_{2t-1} k_2 \leq$$
$$p_{2t} c_{2t} + q_{2t} z_{2t} - q_{2t-1} k_{2t} = p_t \cdot c_t + q_t \cdot z_t - q_{t-1} \cdot k_t = 0$$

for every $(c,z,k) \in T$ and $t \geqq 0$.

Moreover, with such a competitive price system, if, in addition, $p_{20} = 1$ (e.g., $c_{20} > 0$) while $\lim_{t\to\infty} k_{2t} = k_{2\infty} > 0$ (which we already know from section 1 is compatible with efficiency provided $k_{1t} > 0$ for $t \geqq 0$), then

$$\infty > \sum_{t=0}^{\infty} p_t \cdot c_t' = c_{20}' = \bar{k}_2 > \bar{k}_2 - k_{2\infty} = \sum_{t=0}^{\infty} c_{2t} = \sum_{t=0}^{\infty} p_t \cdot c_t$$

for the alternative feasible growth path defined by

$$c_{10}' = h(\bar{k}_1, \bar{k}_2),\ c_{20}' = \bar{k}_2 \text{ and } c_{1t}' = c_{2t}' = k_{1t}' = k_{2t}' = 0 \text{ for } t > 0.$$

This is consistent with (though not entailed by) a second observation about the conjunction of B(9) with an efficient growth path such that: $\lim_{t\to\infty} k_{2t} = k_{2\infty} > 0$, namely, that

$$\lim_{t\to\infty} q_t \cdot k_{t+1} = \lim_{t\to\infty} k_{2t+1} = k_{2\infty} > 0.$$

That is, in both Cases 1 and 2 of the example, the competitive price system described by B(9) may exhibit neither capital-value transversality B(7) nor consumption-value maximization B(8).

In order to establish the rest of the proposition, we utilize the fact that every competitive price system associated with a particular efficient growth path yields (c_t, z_t, k_t) as an optimal solution to the concave programming problem

$$B(10)\begin{cases} \text{maximize} & p_{1t} h(k_1 - z_1, k_2) + p_{2t}(k_2 - z_2) + q_{1t} z_1 + q_{2t} z_2 - q_{1t-1} k_1 - q_{2t-1} k_2 \text{ for } t \geqq 0, \\ \text{subject to} & z_1 \leqq k_1 \qquad \text{(with dual variable } \lambda_{1t} \geqq 0) \\ & z_2 \leqq k_2 \qquad \text{(with dual variable } \lambda_{2t} \geqq 0) \\ \text{and} & \text{nonnegativity} \end{cases}$$

or satisfies that Kuhn-Tucker conditions[20]

$$
B(11)\begin{cases}
z_{1t} \leqq k_{1t} & , = \text{ if } \lambda_{1t} > 0 \quad \text{for } t \geqq 0. \\
z_{2t} \leqq k_{2t} & , = \text{ if } \lambda_{2t} > 0 \\
-p_{1t}h_1(k_{1t}-z_{1t},k_{2t})+q_{1t} \leqq \lambda_{1t} & , = \text{ if } z_{1t} > 0 \\
-p_{2t}+q_{2t} \leqq \lambda_{2t} & , = \text{ if } z_{2t} > 0 \\
p_{1t}h_1(k_{1t}-z_{1t},k_{2t})-q_{1t-1} \leqq -\lambda_{1t} & , = \text{ if } k_{1t} > 0 \\
p_{1t}h_2(k_{1t}-z_{1t},k_{2t})+p_{2t}-q_{2t-1} \leqq -\lambda_{2t} & , = \text{ if } k_{2t} > 0
\end{cases}
$$

It is convenient to analyze the solutions to B(11) by distinguishing two possibilities, depending on whether $\lim_{t\to\infty} k_{2t} = 0$ or $\lim_{t\to\infty} k_{2t} = k_{2\infty} > 0$:

(i) Suppose that $\lim_{t\to\infty} k_{2t} = 0$. In this case, we need only notice that the Kuhn-Tucker conditions B(11) entail the inequalities

$$-p_{1t}h_1+q_{1t} \leqq -p_{1t}h_1+q_{1t-1} \text{ or } q_{1t} \leqq q_{1t-1} \leqq \cdots \leqq q_{1,-1} < \infty$$

and

$$-p_{2t}+q_{2t} \leqq -p_{1t}h_2-p_{2t}+q_{2t-1} \leqq -p_{2t}+q_{2t-1} \text{ or } q_{2t} \leqq q_{2t-1} \leqq \cdots \leqq q_{2,-1} < \infty \text{ for } t \geqq 0.$$

Hence, every competitive price system associated with a particular efficient growth path of this sort satisfies

$$q_t \cdot k_{t+1} \leqq q_{-1} \cdot k_{t+1} \quad \text{for } t \geqq 0,$$

and therefore

$$\lim_{t\to\infty} q_t \cdot k_{t+1} \leqq \lim_{t\to\infty} q_{-1} \cdot k_{t+1} = q_{-1} \cdot \lim_{t\to\infty} k_{t+1} = 0$$

(since by hypothesis $\lim_{t\to\infty} k_{1t} = 0$ while by supposition $\lim_{t\to\infty} k_{2t} = 0$).

(ii) Suppose that $\lim_{t\to\infty} k_{2t} = k_{2\infty} > 0$. In this case, we already know from section 1 that though $\lim_{t\to\infty} k_{1t} = 0$, $k_{1t} > 0$ for $t \geqq 0$. Using this fact in combination with the Kuhn-Tucker conditions, we will show the following: It is always possible to find some competitive price system associated with a particular efficient growth path of this sort which (a) differs from B(9) and also (b) satisfies $\lim_{t\to\infty} q_{2t} = 0$ if and only if we have Case 1 or Case 2a of the example. Since the second of the listed properties entails capital-value transversality,

$$\lim_{t\to\infty} q_t \cdot k_{t+1} = \lim_{t\to\infty}(q_{1,-1}k_{1t+1}+q_{2t}k_{2t+1}) = q_{1,-1}(\lim_{t\to\infty} k_{1t+1})+(\lim_{t\to\infty} q_{2t})(\lim_{t\to\infty} k_{2t+1}) = 0,$$

together with the preceding result, this establishes that it is in these cases, and only in these cases, that an efficient growth path necessarily exhibits consumption-value maximization (at some, but -- referring to the earlier discussion concerning B(9) -- perhaps not all associated competitive price systems).

The argument proceeds by closely examining the structure of the solutions to B(11) in light of the supposition that $k_t > 0$ for $t \geqq 0$: Since generally we know that $z_{it} \begin{Bmatrix} < \\ = \end{Bmatrix} k_{it}$ as $c_{it} \begin{Bmatrix} > \\ = \end{Bmatrix} 0$ for $i = 1,2$, $t \geqq 0$, while specifically we have $k_{it+1} = z_{it} > 0$ ($k_{i0} = \bar{k}_i > 0$) for $i = 1,2$, $t \geqq 0$, after some straightforward logical simplification, the Kuhn-Tucker conditions reduce to

$$B(12)\begin{cases} p_{1t}h_1(k_{1t}-z_{1t},k_{2t}) \geqq q_{1t} & , = \text{ if } c_{1t} > 0 \text{ for } t \geqq 0. \\ p_{2t} \leqq q_{2t} & , = \text{ if } c_{2t} > 0 \\ q_{1t} = q_{1,-1} & \\ q_{2t} = q_{2t-1} - p_{1t}h_2(k_{1t}-z_{1t},k_{2t}) & \end{cases}$$

B(12) tells us immediately that if $q_{1,-1} = 0$, then $q_{1t} = p_{1t} = p_{1t}h_1 = p_{1t}h_2 = 0$ and hence $q_{2t} = q_{2,-1} > 0$ for $t \geqq 0$. Thus, a necessary (and, it turns out, also sufficient) condition for finding some competitive price system which differs from B(9) and also satisfies $\lim\limits_{t\to\infty} q_{2t} = 0$ is finding one with $q_{1,-1} = 1$ (for simplicity, now letting initial capital of type 1 be the numeraire).

In order to establish that only Cases 1 and 2a of the example are consistent with this requirement, suppose that $q_{1,-1} = 1$ and consider the last equation in B(12)

$$B(13) \quad q_{2t} = q_{2t-1} - p_{1t}h_2(k_{1t}-z_{1t},k_{2t})$$

$$= q_{2t-1} - p_{1t}(h(k_{1t}-z_{1t},k_{2t}) - h_1(_{1t}-z_{1t},k_{2t})(k_{1t}-z_{1t})) \text{ for } t \geqq 0$$

(since h is linear homogeneous). If $c_{1t} = 0$, then

$$q_{2t} = q_{2t-1}$$

(since $h = 0$ only if $p_{1t}h_1 = 0$). If $c_{1t} > 0$, then

$$q_{2t} = q_{2t-1} - p_{1t}h_1(k_{1t}-z_{1t},k_{2t})(\frac{h(k_{1t}-z_{1t},k_{2t})}{h_1(k_{1t}-z_{1t},k_{2t})} - (k_{1t}-z_{1t}))$$

(since $h > 0$ only if $(k_{1t}-z_{1t},k_{2t}) > 0$ only if $h_1 > 0$)

$$= q_{2t-1} - \left(\frac{h(k_{1t}-z_{1t},k_{2t})}{h_1(k_{1t}-z_{1t},k_{2t})(k_{1t}-z_{1t})} - 1\right)(k_{1t}-z_{1t})$$

(since, from the first inequality in B(12), if $c_{1t} > 0$, then $p_{1t}h_1 = q_{1t} = q_{1,-1} = 1$). Hence, defining

$$\alpha_t = \begin{cases} 1 & \text{for } c_{1t} = 0 \qquad \text{for } t \geqq 0, \\ \dfrac{h_1(k_{1t}-z_{1t},k_{2t})(k_{1t}-z_{1t})}{h(k_{1t}-z_{1t},k_{2t})} & \text{for } c_{1t} > 0 \end{cases}$$

B(13) can be compactly rewritten as

$$B(14) \quad q_{2t} = q_{2,-1} - \sum_{s=0}^{t}(\frac{1}{\alpha_s} - 1)(k_{1s}-z_{1s}) \text{ for } t \geqq 0.$$

In Case 2b, by employing a construction similar to that utilized in establishing the second part of the proposition in section 2 above, we can find an efficient growth path such that

$$\sum_{s=0}^{\infty}(\frac{1}{\alpha_s} - 1)(k_{1s}-z_{1s}) = \infty .^{21}$$

Thus, for such an efficient growth path, if $q_{1,-1} = 1$, then $\lim\limits_{t\to\infty} q_{2t} = -\infty$, which is "infeasible"; Case 2b of the example may in fact require a "pure inventory" competitive price system.

In Cases 1 and 2a, on the other hand, by picking $0 < \alpha < 1$ such that

$$\frac{h_1(x_1,x_2)x_1}{h(x_1,x_2)} \geqq \alpha \text{ for } 0 \leqq x_1 \leqq \bar{k}_1,\ x_2 \geqq k_{2\infty}$$

we see that

$$0 \leqq \sum_{s=0}^{t} (\frac{1}{\alpha_s} -1)(k_{1s}-z_{1s}) \leqq \sum_{s=0}^{t} (\frac{1}{\alpha} -1)(k_{1s}-z_{1s}) \leqq (\frac{1}{\alpha} -1) \sum_{s=0}^{t} (k_{1s}-k_{1s+1}) =$$

$$(\frac{1}{\alpha} -1)(\bar{k}_1-k_{1t+1}) \leqq (\frac{1}{\alpha} -1)\bar{k}_1 \text{ for } t \geqq 0.$$

Hence, for every efficient growth path such that $\lim_{t\to\infty} k_{2t} = k_{2\infty} > 0$ we can find an associated competitive price system with $q_{1,-1} = 1$. Moreover, by picking

$$q_{2,-1} = \sum_{s=0}^{\infty} (\frac{1}{\alpha_s} -1)(k_{1s}-z_{1s}) < \infty ,$$

so that

$$q_{2t} = \sum_{s=0}^{\infty} (\frac{1}{\alpha_s} -1)(k_{1s}-z_{1s}) - \sum_{s=0}^{t} (\frac{1}{\alpha_s} -1)(k_{1s}-z_{1s})$$

$$= \sum_{s=t+1}^{\infty} (\frac{1}{\alpha_s} -1)(k_{1s}-z_{1s}),$$

we can also insure that this competitive price system satisfies

$\lim_{t\to\infty} q_{2t} = 0.$ $\blacksquare$

d. <u>Another Anomalous Feature of Case 2b</u>: Consider the particular efficient growth path sketched in footnote 21. If we pick consumption goods values

B(15) $\bar{p}_t = (h_1(k_{1t}-z_{1t},\bar{k}_2)^{-1},0)$ for $t \geqq 0$,

then this path is "optimal" in the sense of 3 in the text

B(16) $0 \geqq \limsup_{t\to\infty} \sum_{t=0}^{t} \bar{p}_t \cdot (c'_t-c_t)$ for every (primed) solution to (2),

even though it has no associated competitive price system (4) such that $p_t = \bar{p}_t$ for $t \geqq 0$.

<u>Proof</u>: Since we have already established the latter part of the assertion, all we need establish is B(16). But this follows directly from the fact that an arbitrary feasible growth path satisfies (by monotonicity of h)

$$c'_{1t} = h(k'_{1t}-z'_{1t},k'_{2t}) \leqq h(k'_{1t}-z'_{1t},\bar{k}_2)$$

or (by differentiability and concavity of h)

$$c'_{1t}-c_{1t} \leqq h(k'_{1t}-z'_{1t},\bar{k}_2)-h(k_{1t}-z_{1t},\bar{k}_2) \leqq h_1(k_{1t}-z_{1t},\bar{k}_2)[(k'_{1t}-z'_{1t})-(k_{1t}-z_{1t})]$$

or (by reference to B(15))

$$\bar{p}_t \cdot (c'_t-c_t) = \bar{p}_{1t}(c'_{1t}-c_{1t}) \leqq (k'_{1t}-z'_{1t})-(k_{1t}-z_{1t}) = (k'_{1t}-k'_{1t+1})-(k_{1t}-k_{1t+1})$$

or

$$\sum_{t=0}^{t} \bar{p}_t \cdot (c'_t-c_t) \leqq \sum_{t=0}^{t} [(k'_{1t}-k'_{1t+1})-(k_{1t}-k_{1t+1})]=k_{1t+1}-k'_{1t+1} \text{ for } t \geqq 0.$$

But the last inequality entails that

$$\limsup_{t\to\infty} \sum_{t=0}^{t} \bar{p}_t \cdot (c'_t-c_t) \leqq \limsup_{t\to\infty} (k_{1t+1}-k'_{1t+1})=\lim_{t\to\infty} (k_{1t+1}-k'_{1t+1})= -k'_{1\infty} \leqq 0.$$ $\blacksquare$

FOOTNOTES

1. In every analysis which aims to characterize efficient allocation as competitive allocation (given generalized diminishing returns), a slight gap appears between the statements of necessary and sufficient conditions, a gap occasioned by the fact that every price needn't be positive. For this reason, here and in the balance of the paper we will use "characterize" to mean "are necessary, and also sufficient provided appropriate prices are positive." Also, partly for this reason, in characterizing efficient (or optimal) growth paths, we will concentrate on establishing necessity of various valuation properties. (A more compelling reason for our emphasis is that establishing sufficiency of these same properties is usually a routine matter; see footnote 9 below.)

2. We therefore intentionally defer our discussion of the problems attendant on primary factors until later in the paper (in Sections VII and VIII). It is worth mentioning now, however, that one promising possibility for surmounting these problems is to give explicit consideration to the limitational role of exhaustible resources. Besides displaying our own efforts in this direction (at the end of Appendix A), we strongly recommend looking at the interesting and original work of Mitra [19].

3. It is noteworthy that, generally speaking, in this context the units for measuring consumption goods are interpreted in welfare, or utility terms. Such interpretation is immaterial to our analysis, provided that the static technology available for producing utility and investment goods from capital stocks and primary factors exhibits generalized diminishing returns. However, such interpretation does mean that some care must be taken with the parallel interpretation of the competitive price system associated with an optimal growth path, since, again generally speaking, it could not be used for the purposes of decentralization.

4. With minor modification, several of the results elaborated in the sequel will also carry over to a model in which the commodities and techniques available change over time (in a foreseeable manner). We have chosen not to generalize our model to encompass this sort of nonstationarity in order to avoid cluttering up the presentation.

5. This last maintained assumption is in fact quite restrictive. Later on we effectively remove the force of this restriction by explicitly introducing (short-lived) nonproduced inputs or primary factors (e.g., labor) into the model. Note that (potentially long-lived) nonproduced inputs, both exhaustible resources (e.g., coal) and inexhaustible resources (e.g., land), are already implicitly incorporated into the model: A capital stock, say that of type 1, is an <u>exhaustible resource</u> when $(c,z,k) \varepsilon T$ implies

both $z_1 \leqq k_1$ and $(c,(z_1',z_2,\ldots,z_n), (k_1',k_2,\ldots,k_n)) \in T$ for $z_1' \geqq 0$ and $k_1' - z_1' = k_1 - z_1$, while it is an <u>inexhaustible resource</u> when $(c,z,k) \in T$ implies both $z_1 \leqq k_1$ and $(c,(z_1',z_2,\ldots,z_n), k) \in T$ for $z_1' = k_1$. As mentioned earlier, we will have a good deal more to say about the implications of the presence of exhaustible resources which are also limitational (in Appendix A).

6. For a more complete discussion of the properties (and virtues) of this sort of representation, see Cass-Shell [7]. There is no loss of generality (and some gain in interpretability) in assuming nonnegative output prices. This assertion follows from the observation that, in light of T2, having more investment goods -- and hence capital stocks -- in the present never reduces consumption goods possibilities for the future. (The fact that investment goods prices are nonnegative $q \geqq 0$, however, enters the analysis in a nontrivial way.)

7. In fact, because T is closed and satisfies T4, X is closed in the topology of pointwise convergence (from an argument similar to that found in McFadden [17] on p. 45). Though Malinvaud's original derivation required neither closedness nor any such intrinsic notion of scarcity, the loss in generality sustained from these additional assumptions seems more than compensated by the gain in applicability of our basic separation technique. Moreover, a slight modification of this technique -- substituting the space of finitely-many-nonzero-element sequences with sup norm for the space of summable sequences with sum norm, and thereby avoiding issues of closedness entirely -- can be employed to establish precisely Malinvaud's theorem.

8. After changing units, of course, the sequences x and π would have to be appropriately reinterpreted, $x_0 = \nu_0(\bar{k}_1 - k_{10}),\ldots$ and $\pi_0 = q_{1,-1}/\nu_0,\ldots$. Notice that, given $t^\nu < \infty$, we can always choose the new units for measuring net final outputs or quantities so that $\nu_s = 1$ for $0 \leqq s \leqq t^\nu$. This is a reflection of the fact that it is only the asymptotic behavior of feasible growth paths which creates fundamental difficulties in characterizing efficiency (or optimality).

9. The converse assertion, or sufficiency, follows directly upon specializing (11) to encompass only feasible growth paths,

$$(*) \quad 0 \geqq \sum_{t=0}^{\infty} p_t \cdot (c_t - c_t^*)$$

for every feasible growth path (2) such that

$$\sum_{t=0}^{\infty} \sum_{i=1}^{m} u_{it} |c_{it} - c_{it}^*| < \infty,$$

since (*) is inconsistent with (7) when $p_t > 0$ for $t \geqq 0$. For the corollaries to this proposition, as well as subsequent characterizations, proof of sufficiency is only slightly less direct -- involving reference to (5) and (6) (or, when there are primary factors, (5') and (6')) in order to establish an analogue of (*), since each of these characterizations is couched in terms of an associated competitive price system. Because these arguments are essentially similar and straightforward, we will omit them.

10. That is, for every $t \geq 0$ there exists x^t such that $x^t_s > 0$ for $s = t$ and $x^t_s \geq 0$ for $s \neq t$. This claim follows from two facts, first, as we have already seen, that $x^0 = x^*$ works, and second, since $\bar{k} > 0$ and T5 obtains, that the economy is capable of producing positive investment goods output in any given period, i.e., that x^t for $1 \leq t \leq n$ can be generated by the trivial growth path, while x^t for $t \geq n + 1$ can be generated a growth path involving only pure capital accumulation up through period t.

11. The former assertion follows from the observation that the structure of the set $\{(c,z,k) : (c,z,k,\bar{\ell}) \in T'\}$ contains all the information we need to know about the structure of T' itself, and that this set can be equally well viewed as the projection of a cross-section from 1-higher dimension as from h-higher dimensions. (See also the comment concerning T'3 below.) The latter assertion follows from the observation that setting $\bar{\ell} = 1$ amounts to defining the unit for measuring the primary factor.

12. To go from T'5 to T'5' take

$(c'',z'',k'',\ell'') = \alpha(c',z',k',\ell)$ for some $0 < \alpha < 1$ such that $\alpha(c',z') > (c,z)$,

and from T'5' to T'5

$(c',z') = (1-\beta)(c,z) + \beta(c'',z'')$ from some $0 < \beta < 1$ such that $(1-\beta)k + \beta k'' \leq k'$ (and hence $(1-\beta)\ell + \beta\,\ell'' < \ell$).

13. An alternative method of achieving the same result is simply to reinterpret the technology $T = \{(c,z,k) : (c,z,k,1) \in T'\}$ and define the imputation $w_t = p_t \cdot c^*_t - q_t \cdot z^*_t - q_{t-1} \cdot k^*_t$. Then (4') follows from the left-hand inequality in (4) -- which is demonstrable without reference to T3 -- by virtue of T'2 and T'3. However, while this maneuver does reflect the intrinsic economics of residual payment to a fixed factor, unlike the argument sketched in the text, it does not yield any additional information about value maximization (see the following discussion).

14. Similarly, in the Duality corollary to the Capital-Value Transversality theorem, the dual problem (28) becomes

(28') minimize $q_{-1} \cdot \bar{k} + \sum_{t=0}^{\infty} w_t$ subject to (3') and $p_t = \bar{p}_t$ for $t \geq 0$.

15. Here, of course, we are implicitly accepting the common convention that a maximum must be finite. This makes good sense in this instance, since otherwise consumption-value maximization may be essentially vacuous (as it is, for instance, again in the canonical

one-good model, with any given consumption goods values such that $\inf_{s \geq 0} \bar{p}_{t_s} > 0$ for some subsequence of periods $\{t_s\}$).

The example in Appendix B shows that, even without primary factors, the notion of consumption-value maximization may be too limiting. Indeed, generally speaking, in order that the problem (27), or better yet, the problem

(27') maximize $\sum_{t=0}^{\infty} \bar{p}_t \cdot c_t$ subject to (2')

have a solution, the permissible choice of consumption goods values (20) will typically exclude perfectly acceptable growth paths (i.e., some efficient growth paths) from serious consideration. (In this sense, the class of growth models considered in Section V appears to be quite special.)

16. Obviously, after N is redefined to be compatible with T'

(15') $N = \{(c,y) : y = z - k \text{ and } (c,z,k,\ell) \in T' \text{ for some } \ell \geq 0\}$.

Then the proof of this equivalency is identical to that of Lemma 1A but for the substitution of (c^s,z^s,k^s,ℓ^s) for (c^s,z^s,k^s), (c,z,k,ℓ) for (c,z,k) and T' for T.

17. The specific counterexample detailed at the end of Appendix B clearly shows that, for the purpose of characterizing weaker notions of optimality than consumption-value maximization, something like this substitution condition cannot be dispensed with. Presently the assumptions $\bar{k} > 0$, $\bar{\ell}_t = 1$ for $t \geq 0$, T'5 and T'8 play (and previously, in Section VI, the assumptions $\bar{k} > 0$ and T5 played) precisely the same role as does Slater's condition (or, more generally, some version of "constraint qualification") in ordinary concave programming.

18. Because of T'5 (given $\bar{k} > 0$ and $\bar{\ell}_t = 1$ for $t \geq 0$) and (20) we know that $c^*_t \geq 0$ for some $t \geq 0$. Suppose that $c^*_s = 0$ for $0 \leq s < t^* < \infty$ and $c^*_{t^*} \geq 0$. Appealing to T'8, pick $0 \leq c' \leq c^*_{t^*}$ and $z' > z^*_{t^*}$ such that $(c',z',k^*_{t^*},1) \in T'$. Then the growth path $(c_s,z_s,k_s,\ell_s) = \alpha(c^*_s,z^*_s,k^*_s,\ell^*_s) + (1-\alpha)(0,0,0,1)$ for $0 \leq s < t^*$, $(c_s,z_s,k_s,\ell_s) = \alpha(c',z',k^*_s,1) + (1-\alpha)(0,0,0,1)$ for $s = t^*$, $(c_s,z_s,k_s,\ell_s) = (c^*_s,z^*_s,\alpha z',1)$ for $s = t^*+1$ and $(c_s,z_s,k_s,\ell_s) = (c^*_s,z^*_s,k^*_s,\ell^*_s)$ for $s > t^*+1$ is feasible from initial capital stocks $0 \leq \alpha\, k^*_0 < \bar{k}$ provided only that $0 < \alpha < 1$ and $\alpha z' \geq z^*_{t^*}$. Now appealing to T'5, these "surplus" initial capital stocks $\bar{k} - \alpha k^*_0 > 0$ can be used to generate growth paths which yield excess investment goods output $z_t - k_{t+1} > 0$ or excess primary factor input $1 - \ell_t > 0$ in any given period t. These paths in turn generate the requisite quantity sequences x^t for $t \geq 1$.

19. The closed, multi-sector model is the specialization of our general model in which $m = n$ and $T = \{(c,z,y) : c + z = y,$ $(c,z) \geq 0$ and $(y,k) \in S\}$, where $S = \{(y,k)\}$ is the static techology for producing (undifferentiated) outputs from capital stock inputs.

20. B(10) obviously satisfies Slater's condition (since $(z,k) = (0,0,1,1)$ is a feasible solution). Thus, the conditions B(11) are both necessary and sufficient. It almost goes without saying that in Case 2 these particular Kuhn-Tucker conditions only make sense provided that

$$p_{1t} = p_{1t}h_1(k_{1t}-z_{1t},k_{2t}) = p_{1t}h_2(k_{1t}-z_{1t},k_{2t}) = 0$$

whenever $c_{1t} = h(k_{1t}-z_{1t},k_{2t}) = 0$ and hence $k_{1t}-z_{1t} = 0$.

21. Specifically, referring to the earlier construction, now let the subsequence of periods $\{t_s\}$ be such that

$$t_0 = 0$$

and

$$t_{s+1} = \min\{t : t>t_s \text{ and } \left(\frac{h\left(\frac{(1-\alpha)\bar{k}_1\alpha^s}{t-t_s},\bar{k}_2\right)}{h_1\left(\frac{(1-\alpha)\bar{k}_1\alpha^s}{t-t_s},\ \bar{k}_2\right)\left(\frac{(1-\alpha)\bar{k}_1\alpha^s}{t-t_s}\right)} -1\right)(1-\alpha)\bar{k}_1\alpha^s \geqq \beta\} \text{ for } s\geqq 0$$

(which is legitimate since Case 2b is easily shown to be characterized by the property that $\lim_{x_1\to 0^+} \frac{h_1(x_1,\bar{k}_2)x_1}{h(x_1,\bar{k}_2)} = 0$ for $\bar{k}_2 > 0$).

Then, the rest of the argument is virtually unchanged from before, since it simply involves calculating the lower bound

$$\sum_{t=0}^{t_{s+1}-1} (\frac{1}{\alpha_t} -1)(k_{1t}-z_{1t}) = \ldots \geqq \beta(s+1) \text{ for } s \geqq 0.$$

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